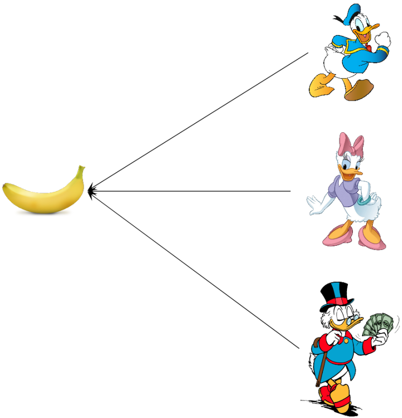
Selling Bananas in an Uncertain Environment

Vasilis Livanos

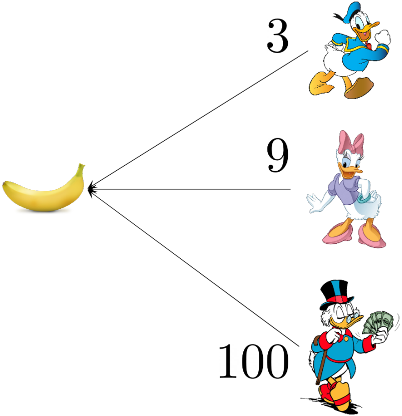
University of Illinois Urbana-Champaign

November 29th, 2023

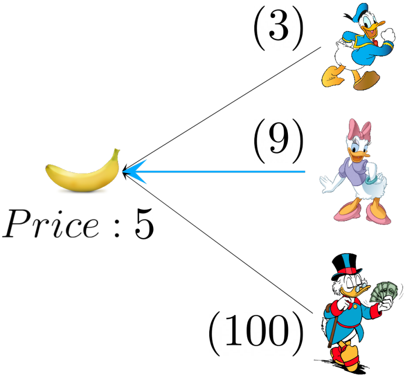
Auction Design



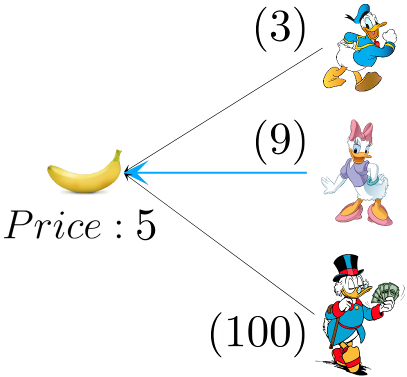
Auction Design



Auction Design



Auction Design



How to set the price?

Overview

1. **Distribution-optimal prophet inequalities** [L., Mehta '22, L. '23]
   - Unified proof for both max and min I.I.D prophet inequality
   - Competition complexity
2. **Oracle-augmented prophet inequalities** [Har-Peled, Harb, L. '23]
   - Connection with top-1-of-k model
   - Upper-lower bounds for I.I.D. case
   - Upper-lower bounds for general case (adversarial order)
3. **Optimal greedy OCRSs** [L., '22]
   - $1/e$-selectable greedy OCRS for single-item
   - $1/e$ hardness
   - Extension to transversal matroids
4. **Submodular prophet inequalities** [Chekuri, L. '21]
   - Small constant SPI via OCRS
   - Generalized framework for several constraints
   - Correlation gap

Overview

1. **Distribution-optimal prophet inequalities** [L., Mehta '22, L. '23]
   - Unified proof for both max and min I.I.D prophet inequality
   - Competition complexity
2. **Oracle-augmented prophet inequalities** [Har-Peled, Harb, L. '23]
   - Connection with top-1-of-k model
   - Upper-lower bounds for I.I.D. case
   - Upper-lower bounds for general case (adversarial order)
3. **Optimal greedy OCRSs** [L., '22]
   - $1/e$-selectable greedy OCRS for single-item
   - $1/e$ hardness
   - Extension to transversal matroids
4. **Submodular prophet inequalities** [Chekuri, L. '21]
   - Small constant SPI via OCRS
   - Generalized framework for several constraints
   - Correlation gap

Optimal Stopping: The Prophet Inequality

[Krengel, Sucheston, Garling '77]

$X_1, X_2, \ldots, X_n \sim$ (known) $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$
arrive in *adversarial* order.

- Design *stopping time* to maximize selected value.
- Compare against all-knowing *prophet*: $\mathbb{E}[\max_i X_i]$.

$$\mathcal{U}[13, 14]$$

$$\mathcal{U}[7, 16]$$

$$\mathcal{U}[0, 20]$$

$$\begin{cases} 1000 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{U}[13, 14]$	$\mathcal{U}[7, 16]$	$\mathcal{U}[0, 20]$	$\begin{cases} 1000 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$

$X_1 = 13.74$

$\mathcal{U}[13, 14]$	$\mathcal{U}[7, 16]$	$\mathcal{U}[0, 20]$	$\begin{cases} 1000 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$
$X_1 = 13.74$	$X_2 = 15.66$		

$\mathcal{U}[13, 14]$	$\mathcal{U}[7, 16]$	$\mathcal{U}[0, 20]$	$\begin{cases} 1000 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$
$X_1 = 13.74$	$X_2 = 15.66$	$X_3 = 16.67$	

$\mathcal{U}[13, 14]$	$\mathcal{U}[7, 16]$	$\mathcal{U}[0, 20]$	$\begin{cases} 1000 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$
$X_1 = 13.74$	$X_2 = 15.66$	$X_3 = 16.67$	$X_4 = 0$

$\mathbb{E}[\max\{X_1, X_2, X_3, X_4\}] \approx 24.66$

$\mathbb{E}[OPTALG\{X_1, X_2, X_3, X_4\}] \approx 13.37$

Optimal strategy was to select X_1.

Prophet Inequality [Krengel, Sucheston, Garling '77, '78]

$\exists$ stopping strategy that achieves $1/2 \cdot \mathbb{E}[\max_i X_i]$,
and this is tight.

$$X_1 = 1 \quad \text{w.p. } 1, \text{ and } X_2 = \begin{cases} 1/\varepsilon & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1-\varepsilon \end{cases}$$

$\mathbb{E}$ [ALG] $= 1$ for all algorithms.

$\mathbb{E}[\max\{X_1, X_2\}] = \frac{1}{\varepsilon} \cdot \varepsilon + 1 \cdot (1-\varepsilon) = 2 - \varepsilon$.

- $\frac{\mathbb{E}[ALG]}{\mathbb{E}[\max_i X_i]}$: Competitive Ratio

Prophet Inequality [Krengel, Sucheston, Garling '77, '78]

∃ stopping strategy that achieves $1/2 \cdot \mathbb{E}[\max_i X_i]$,
and this is tight.

$$X_1 = 1 \quad \text{w.p. } 1, \text{ and } X_2 = \begin{cases} 1/\varepsilon & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1 - \varepsilon \end{cases}$$

$\mathbb{E}\,[\text{ALG}] = 1$ for all algorithms.

$\mathbb{E}[\max\{X_1, X_2\}] = \frac{1}{\varepsilon} \cdot \varepsilon + 1 \cdot (1 - \varepsilon) = 2 - \varepsilon$.

- $\frac{\mathbb{E}[ALG]}{\mathbb{E}[\max_i X_i]}$: Competitive Ratio
- Idea: Set *threshold* T, accept first $X_i \geq T$.
  T = price in auction.

Prophet Inequality [Krengel, Sucheston, Garling '77, '78]

$\exists$ stopping strategy that achieves $1/2 \cdot \mathbb{E}[\max_i X_i]$,
and this is tight.

$$X_1 = 1 \quad \text{w.p. } 1, \text{ and } X_2 = \begin{cases} 1/\varepsilon & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1-\varepsilon \end{cases}$$

$\mathbb{E}\,[\text{ALG}] = 1$ for all algorithms.

$\mathbb{E}[\max\{X_1, X_2\}] = \frac{1}{\varepsilon} \cdot \varepsilon + 1 \cdot (1-\varepsilon) = 2 - \varepsilon$.

- $\frac{\mathbb{E}[ALG]}{\mathbb{E}[\max_i X_i]}$: Competitive Ratio
- Idea: Set *threshold* T, accept first $X_i \geq T$.
  T = price in auction.
  [Samuel-Cahn '84]

Prophet Inequality [Krengel, Sucheston, Garling '77, '78]

$\exists$ stopping strategy that achieves $1/2 \cdot \mathbb{E}[\max_i X_i]$,
and this is tight.

$$X_1 = 1 \quad \text{w.p. } 1, \text{ and } X_2 = \begin{cases} 1/\varepsilon & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1 - \varepsilon \end{cases}$$

$\mathbb{E}\,[\text{ALG}] = 1$ for all algorithms.

$\mathbb{E}[\max\{X_1, X_2\}] = \frac{1}{\varepsilon} \cdot \varepsilon + 1 \cdot (1 - \varepsilon) = 2 - \varepsilon$.

- $\frac{\mathbb{E}[ALG]}{\mathbb{E}[\max_i X_i]}$: Competitive Ratio
- Idea: Set *threshold* T, accept first $X_i \geq T$.
  T = price in auction.
  [Samuel-Cahn '84]
  [Kleinberg, Weinberg '12]

Two proofs in one?? 

For any T,

$$\mathbb{E}[ALG] \geq \Pr[\max_i X_i \geq T]\, T + \sum_i \Pr[\text{We reach } i]\; \mathbb{E}[\max\{X_i - T, 0\}]$$

Two proofs in one?? 

For any T,

$$\mathbb{E}[ALG] \geq \Pr[\max_i X_i \geq T]\, T + \sum_i \Pr[\text{We reach } i]\ \mathbb{E}[\max\{X_i - T, 0\}]$$

$$\geq \Pr[\max_i X_i \geq T]\, T + \Pr[\max_i X_i < T]\ \mathbb{E}\left[\sum_i X_i - T\right]$$

Two proofs in one?? 

For any T,

$$\begin{aligned}
\mathbb{E}[ALG] &\geq \Pr[\max_i X_i \geq T]\, T + \sum_i \Pr[\text{We reach } i]\ \mathbb{E}[\max\{X_i - T, 0\}] \\
&\geq \Pr[\max_i X_i \geq T]\, T + \Pr[\max_i X_i < T]\ \mathbb{E}\left[\sum_i X_i - T\right] \\
&\geq \Pr[\max_i X_i \geq T]\, T + (1 - \Pr[\max_i X_i \geq T]) \left(\mathbb{E}\left[\max_i X_i\right] - T\right).
\end{aligned}$$

Two proofs in one?? 

For any T,

$$\mathbb{E}[ALG] \geq \Pr[\max_i X_i \geq T]\, T + \sum_i \Pr[\text{We reach } i]\; \mathbb{E}[\max\{X_i - T, 0\}]$$

$$\geq \Pr[\max_i X_i \geq T]\, T + \Pr[\max_i X_i < T]\; \mathbb{E}\left[\sum_i X_i - T\right]$$

$$\geq \Pr[\max_i X_i \geq T]\, T + (1 - \Pr[\max_i X_i \geq T]) \left(\mathbb{E}\left[\max_i X_i\right] - T\right).$$

- [Samuel-Cahn '84]: Set T such that $\Pr[\max_i X_i \geq T] = 1/2$.

$$\mathbb{E}[ALG] \geq \frac{1}{2}\, T + \frac{1}{2} \left(\mathbb{E}\left[\max_i X_i\right] - T\right) = \frac{\mathbb{E}[\max_i X_i]}{2}.$$

Two proofs in one?? 

For any T,

$$\mathbb{E}[ALG] \geq \Pr[\max_i X_i \geq T]\, T + \sum_i \Pr[\text{We reach } i]\ \mathbb{E}[\max\{X_i - T, 0\}]$$

$$\geq \Pr[\max_i X_i \geq T]\, T + \Pr[\max_i X_i < T]\ \mathbb{E}\left[\sum_i X_i - T\right]$$

$$\geq \Pr[\max_i X_i \geq T]\, T + (1 - \Pr[\max_i X_i \geq T]) \left(\mathbb{E}\left[\max_i X_i\right] - T\right).$$

- [Samuel-Cahn '84]: Set T such that $\Pr[\max_i X_i \geq T] = 1/2$.

$$\mathbb{E}[ALG] \geq \frac{1}{2} T + \frac{1}{2}\left(\mathbb{E}\left[\max_i X_i\right] - T\right) = \frac{\mathbb{E}[\max_i X_i]}{2}.$$

- [Kleinberg, Weinberg '12]: Set $T = \frac{1}{2} \cdot \mathbb{E}[\max_i X_i]$.

$$\mathbb{E}[ALG] \geq \Pr[\max_i X_i \geq T]\, \frac{\mathbb{E}[\max_i X_i]}{2} + \left(1 - \Pr[\max_i X_i \geq T]\right) \cdot$$
$$\cdot \left(\mathbb{E}\left[\max_i X_i\right] - \frac{1}{2} \cdot \mathbb{E}[\max_i X_i]\right)$$

Two proofs in one?? 

For any T,

$$\mathbb{E}[ALG] \geq \Pr[\max_i X_i \geq T]\, T + \sum_i \Pr[\text{We reach } i]\; \mathbb{E}[\max\{X_i - T, 0\}]$$

$$\geq \Pr[\max_i X_i \geq T]\, T + \Pr[\max_i X_i < T]\; \mathbb{E}\left[\sum_i X_i - T\right]$$

$$\geq \Pr[\max_i X_i \geq T]\, T + (1 - \Pr[\max_i X_i \geq T]) \left(\mathbb{E}\left[\max_i X_i\right] - T\right).$$

- [Samuel-Cahn '84]: Set T such that $\Pr[\max_i X_i \geq T] = 1/2$.

$$\mathbb{E}[ALG] \geq \frac{1}{2} T + \frac{1}{2}\left(\mathbb{E}\left[\max_i X_i\right] - T\right) = \frac{\mathbb{E}[\max_i X_i]}{2}.$$

- [Kleinberg, Weinberg '12]: Set $T = \frac{1}{2} \cdot \mathbb{E}[\max_i X_i]$.

$$\mathbb{E}[ALG] \geq \Pr[\max_i X_i \geq T]\, \frac{\mathbb{E}[\max_i X_i]}{2} + \left(1 - \Pr[\max_i X_i \geq T]\right) \frac{\mathbb{E}[\max_i X_i]}{2}$$

Two proofs in one?? 

For any T,

$$\mathbb{E}[ALG] \geq \Pr[\max_i X_i \geq T]\, T + \sum_i \Pr[\text{We reach } i]\ \mathbb{E}[\max\{X_i - T, 0\}]$$

$$\geq \Pr[\max_i X_i \geq T]\, T + \Pr[\max_i X_i < T]\ \mathbb{E}\left[\sum_i X_i - T\right]$$

$$\geq \Pr[\max_i X_i \geq T]\, T + (1 - \Pr[\max_i X_i \geq T])\left(\mathbb{E}\left[\max_i X_i\right] - T\right).$$

- [Samuel-Cahn '84]: Set T such that $\Pr[\max_i X_i \geq T] = 1/2$.

$$\mathbb{E}[ALG] \geq \frac{1}{2} T + \frac{1}{2}\left(\mathbb{E}\left[\max_i X_i\right] - T\right) = \frac{\mathbb{E}[\max_i X_i]}{2}.$$

- [Kleinberg, Weinberg '12]: Set $T = \frac{1}{2} \cdot \mathbb{E}[\max_i X_i]$.

$$\mathbb{E}[ALG] \geq \Pr[\max_i X_i \geq T]\, \frac{\mathbb{E}[\max_i X_i]}{2} + \left(1 - \Pr[\max_i X_i \geq T]\right) \frac{\mathbb{E}[\max_i X_i]}{2}$$

$$= \frac{\mathbb{E}[\max_i X_i]}{2}.$$

Procurement $\implies$ Minimization

Procurement $\implies$ Minimization

What if objective is $\min_i X_i$? Same problem?

- Objective: Minimize selected value, compare against $\underline{\mathbb{E}[\min_i X_i]}$.
  Forced to select an element.

Procurement $\implies$ Minimization

What if objective is $\min_i X_i$? Same problem?

- Objective: Minimize selected value, compare against $\mathbb{E}[\min_i X_i]$.
  Forced to select an element.
- No bound on competitive ratio for general (non-I.I.D.) distributions.
  [Esfandiari, Hajiaghayi, Liaghat, Monemizadeh '15]
- What about I.I.D.? $(\mathcal{D}_1 = \mathcal{D}_2 = \cdots = \mathcal{D}_n = \mathcal{D})$

Procurement $\implies$ Minimization

What if objective is $\min_i X_i$? Same problem?

- Objective: Minimize selected value, compare against $\mathbb{E}[\min_i X_i]$.
  Forced to select an element.
- No bound on competitive ratio for general (non-I.I.D.) distributions.
  [Esfandiari, Hajiaghayi, Liaghat, Monemizadeh '15]
- What about I.I.D.? $(\mathcal{D}_1 = \mathcal{D}_2 = \cdots = \mathcal{D}_n = \mathcal{D})$

  No hope for universal bound: [Lucier '22]

  $\mathcal{D} : F(x) = 1 - 1/x$, with $x \in [1, +\infty)$ (Equal-revenue distribution).

  $\mathbb{E}[X] = 1 + \int_1^\infty (1 - F(x))\, dx = +\infty$, but

  $\mathbb{E}[\min\{X_1, X_2\}] = 1 + \int_1^\infty (1 - F(x))^2\, dx < +\infty$.

I.I.D. Prophet Inequality

I.I.D. Prophet Inequality

I.I.D. Prophet Inequality [Hill, Kertz '82, Correa, Foncea, Hoeksma, Oosterwijk, Vredeveld '21]]

For any $\mathcal{D}$, $\exists$ threshold stopping strategy $\tau_1, \tau_2, \ldots, \tau_n$ that achieves $\beta \cdot \mathbb{E}[\max_i X_i]$, where $\beta \approx 0.745$, and this is tight.

I.I.D. Prophet Inequality

I.I.D. Prophet Inequality [Hill, Kertz '82, Correa, Foncea, Hoeksma, Oosterwijk, Vredeveld '21]]

For any $\mathcal{D}$, $\exists$ threshold stopping strategy $\tau_1, \tau_2, \ldots, \tau_n$ that achieves $\beta \cdot \mathbb{E}[\max_i X_i]$, where $\beta \approx 0.745$, and this is tight.

[Hill, Kertz '82]]

For any $\mathcal{D}$, $\exists$ a single threshold τ such that selecting the first $X_i \geq \tau$ achieves $(1 - 1/e) \cdot \mathbb{E}[\max_i X_i] \approx 0.632 \cdot \mathbb{E}[\max_i X_i]$.

I.I.D. Prophet Inequality

I.I.D. Prophet Inequality [Hill, Kertz '82, Correa, Foncea, Hoeksma, Oosterwijk, Vredeveld '21]]

For any $\mathcal{D}$, $\exists$ threshold stopping strategy $\tau_1, \tau_2, \ldots, \tau_n$ that achieves $\beta \cdot \mathbb{E}[\max_i X_i]$, where $\beta \approx 0.745$, and this is tight.

[Hill, Kertz '82]]

For any $\mathcal{D}$, $\exists$ a single threshold τ such that selecting the first $X_i \geq \tau$ achieves $(1 - 1/e) \cdot \mathbb{E}[\max_i X_i] \approx 0.632 \cdot \mathbb{E}[\max_i X_i]$.

- Minimization?

Intuition:

Set $T = c \cdot \mathbb{E}[\min_i X_i]$.

I.I.D. Prophet Inequality [Hill, Kertz '82, Correa, Foncea, Hoeksma, Oosterwijk, Vredeveld '21]]

For any $\mathcal{D}$, $\exists$ threshold stopping strategy $\tau_1, \tau_2, \ldots, \tau_n$ that achieves $\beta \cdot \mathbb{E}[\max_i X_i]$, where $\beta \approx 0.745$, and this is tight.

[Hill, Kertz '82]]

For any $\mathcal{D}$, $\exists$ a single threshold τ such that selecting the first $X_i \geq \tau$ achieves $(1 - 1/e) \cdot \mathbb{E}[\max_i X_i] \approx 0.632 \cdot \mathbb{E}[\max_i X_i]$.

- Minimization?

~~Intuition~~ False Intuition:

Doesn't work! $\Pr[\text{We are forced to select } X_n] \geq c$.

Towards a Unified Analysis

Towards a Unified Analysis

Optimal policy: Set $\tau_i = \mathbb{E}[\mathrm{OPTALG}_{i+1,\ldots,n}]$, accept first $X_i \leq \tau_i$.
How to analyze it?

Towards a Unified Analysis

Optimal policy: Set $\tau_i = \mathbb{E}[\mathrm{OPTALG}_{i+1,\dots,n}]$, accept first $X_i \le \tau_i$.
How to analyze it?

Worst-case instance for MAX: $n \to \infty \implies$ Fix $\mathcal{D}$ and take $n \to \infty$.

Asymptotic Competitive Ratio (ACR)

$$\lambda_{min} = \lim_{n\to\infty} \frac{\mathbb{E}[ALG(n)]}{\mathbb{E}[\min_{i=1}^{n} X_i]}.$$

Towards a Unified Analysis

Optimal policy: Set $\tau_i = \mathbb{E}[\text{OPTALG}_{i+1,\dots,n}]$, accept first $X_i \le \tau_i$.
How to analyze it?

Worst-case instance for Max: $n \to \infty \implies$ Fix $\mathcal{D}$ and take $n \to \infty$.

Asymptotic Competitive Ratio (ACR)

$$\lambda_{min} = \lim_{n\to\infty} \frac{\mathbb{E}[ALG(n)]}{\mathbb{E}[\min_{i=1}^{n} X_i]}. \qquad \lambda_{max} = \lim_{n\to\infty} \frac{\mathbb{E}[ALG(n)]}{\mathbb{E}[\max_{i=1}^{n} X_i]}.$$

Towards a Unified Analysis

Optimal policy: Set $\tau_i = \mathbb{E}[\mathrm{OPTALG}_{i+1,\ldots,n}]$, accept first $X_i \leq \tau_i$.
How to analyze it?

Worst-case instance for MAX: $n \to \infty \implies$ Fix $\mathcal{D}$ and take $n \to \infty$.

Asymptotic Competitive Ratio (ACR)

$$\lambda_{min} = \lim_{n\to\infty} \frac{\mathbb{E}[ALG(n)]}{\mathbb{E}[\min_{i=1}^n X_i]}. \qquad \lambda_{max} = \lim_{n\to\infty} \frac{\mathbb{E}[ALG(n)]}{\mathbb{E}[\max_{i=1}^n X_i]}.$$

- $M_n = \max\{X_1, \ldots, X_n\}$
- $m_n = \min\{X_1, \ldots, X_n\}$
- Distribution of M_n, m_n as $n \to \infty$?

Towards a Unified Analysis

Optimal policy: Set $\tau_i = \mathbb{E}[\mathrm{OPTALG}_{i+1,\dots,n}]$, accept first $X_i \le \tau_i$.
How to analyze it?

Worst-case instance for MAX: $n \to \infty \implies$ Fix $\mathcal{D}$ and take $n \to \infty$.

Asymptotic Competitive Ratio (ACR)

$$\lambda_{min} = \lim_{n\to\infty} \frac{\mathbb{E}[ALG(n)]}{\mathbb{E}[\min_{i=1}^{n} X_i]}. \qquad \lambda_{max} = \lim_{n\to\infty} \frac{\mathbb{E}[ALG(n)]}{\mathbb{E}[\max_{i=1}^{n} X_i]}.$$

- $M_n = \max\{X_1, \dots, X_n\}$
- $m_n = \min\{X_1, \dots, X_n\}$
- Distribution of M_n, m_n as $n \to \infty$?
- $\lim_{n\to\infty} M_n = +\infty, \quad \lim_{n\to\infty} m_n = 0 \implies$ Re-scaling

Technique: Extreme Value Theory

Extreme Value Theorem [Fisher, Tippett '28, Gnedenko '43]

Assume there exist sequences $a_n > 0, b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} F_{M_n}(a_n x + b_n) = G_\gamma^+(x).$$

Then,

$$G_\gamma^+(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), & \text{if } \gamma \neq 0 \\ \exp\left(-\exp\left(-x\right)\right), & \text{if } \gamma = 0 \end{cases}.$$

Technique: Extreme Value Theory

Extreme Value Theorem [Fisher, Tippett '28, Gnedenko '43]

Assume there exist sequences $a_n > 0, b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} F_{M_n}(a_n x + b_n) = G_\gamma^+(x).$$

Then,

$$G_\gamma^+(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), & \text{if } \gamma \neq 0 \\ \exp\left(-\exp\left(-x\right)\right), & \text{if } \gamma = 0 \end{cases}.$$

- G : Extreme Value Distribution, γ : Extreme Value Index
- Three distinct G_γ^+'s:
  - $\gamma < 0$: Reverse Weibull
  - $\gamma = 0$: Gumbel
  - $\gamma > 0$: Fréchet

Technique: Extreme Value Theory

Extreme Value Theorem [Fisher, Tippett '28, Gnedenko '43]

Assume there exist sequences $a_n > 0, b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} F_{M_n}(a_n x + b_n) = G_\gamma^+(x).$$

Then,

$$G_\gamma^+(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), & \text{if } \gamma \neq 0 \\ \exp\left(-\exp\left(-x\right)\right), & \text{if } \gamma = 0 \end{cases}.$$

- G : Extreme Value Distribution, γ : Extreme Value Index
- Three distinct G_γ^+'s: $\gamma < 0, \quad \gamma = 0, \quad \gamma > 0$
- Central Limit Theorem analogue for Max.
- Can get similar result for Min, but γ changes.

Technique: Extreme Value Theory

Extreme Value Theorem [Fisher, Tippett '28, Gnedenko '43]
Assume there exist sequences $a_n > 0, b_n \in \mathbb{R}$ such that

$$\lim_{n \to \infty} F_{M_n}(a_n x + b_n) = G_\gamma^+(x).$$

Then,

$$G_\gamma^+(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), & \text{if } \gamma \neq 0 \\ \exp\left(-\exp\left(-x\right)\right), & \text{if } \gamma = 0 \end{cases}.$$

- G : Extreme Value Distribution, γ : Extreme Value Index
- Three distinct G_γ^+'s: $\gamma < 0, \quad \gamma = 0, \quad \gamma > 0$
- Central Limit Theorem analogue for Max.
- Can get similar result for Min, but γ changes.
- Conditions $\implies \mathcal{D}$ follows EVT.

IID PI via Extreme Value Theory

$\Gamma(x) = (x-1)!$

Theorem [L., Mehta '22, L. '23]

Assume there exist sequences $a_n > 0, b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} F_{M_n}(a_n x + b_n) = G_\gamma^+(x)$$

for some γ

$$\lim_{n\to\infty} F_{m_n}(a_n x + b_n) = G_\gamma^-(x)$$

for some γ

Then, the optimal DP achieves a competitive ratio, as $n \to \infty$, of

$$ACR_{Max} = \min\left\{\frac{(1-\gamma)^{-\gamma}}{\Gamma(1-\gamma)}, 1\right\}.$$

$$ACR_{Min} = \max\left\{\frac{(1-\gamma)^{-\gamma}}{\Gamma(1-\gamma)}, 1\right\}.$$

IID PI via Extreme Value Theory

$\Gamma(x) = (x-1)!$

Theorem [L., Mehta '22, L. '23]

Assume there exist sequences $a_n > 0, b_n \in \mathbb{R}$ such that

$\lim_{n\to\infty} F_{M_n}(a_n x + b_n) = G_\gamma^+(x)$	$\lim_{n\to\infty} F_{m_n}(a_n x + b_n) = G_\gamma^-(x)$
for some γ	for some γ

Then, the optimal DP achieves a competitive ratio, as $n \to \infty$, of

$ACR_{Max} = \min\left\{\frac{(1-\gamma)^{-\gamma}}{\Gamma(1-\gamma)}, 1\right\}.$	$ACR_{Min} = \max\left\{\frac{(1-\gamma)^{-\gamma}}{\Gamma(1-\gamma)}, 1\right\}.$

- Distribution-optimal closed form!
- Unified analysis of competitive ratio for both Max and Min.

IID PI via Extreme Value Theory

$\Gamma(x) = (x-1)!$

Theorem [L., Mehta '22, L. '23]

Assume there exist sequences $a_n > 0, b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} F_{M_n}(a_n x + b_n) = G_\gamma^+(x)$$

for some γ

$$\lim_{n\to\infty} F_{m_n}(a_n x + b_n) = G_\gamma^-(x)$$

for some γ

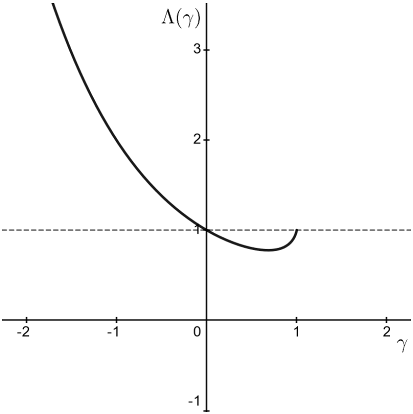
Then, the optimal DP achieves a competitive ratio, as $n \to \infty$, of

$$ACR_{Max} = \min\left\{\frac{(1-\gamma)^{-\gamma}}{\Gamma(1-\gamma)}, 1\right\}.$$

$$ACR_{Min} = \max\left\{\frac{(1-\gamma)^{-\gamma}}{\Gamma(1-\gamma)}, 1\right\}.$$

- Distribution-optimal closed form!
- Unified analysis of competitive ratio for both Max and Min.
- $\mathcal{D}$ MHR $\implies$ $ACR_{Max} = 1$ & $ACR_{Min} \leq 2$

Asymptotic Competitive Ratio



For $\gamma \to -\infty$, by Stirling's approximation

$$\frac{(1-\gamma)^{-\gamma}}{\Gamma(1-\gamma)} \approx e^{-\gamma}.$$

Asymptotic Competitive Ratio

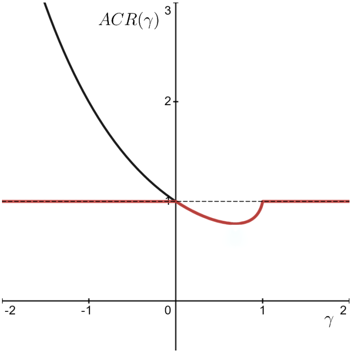


Figure: ACR(γ) for MAX

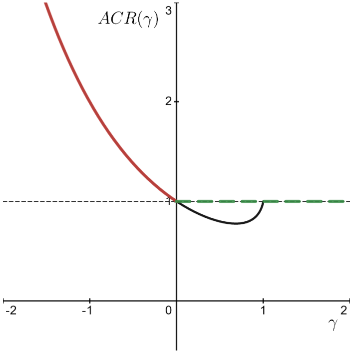


Figure: ACR(γ) for MIN

- "$\mathcal{D}$ follows EVT": most general class we expect closed-form.

High-Level Approach

$F(t) = \Pr_{X \sim \mathcal{D}}[X \leq t], \quad F^{\leftarrow}(p)$: inverse of F ("Quantile function").

Using EVT and heavy-machinery from theory of regularly-varying functions:

<u>Max</u>	<u>Min</u>

High-Level Approach

$F(t) = \Pr_{X\sim\mathcal{D}}[X \le t]$, $\quad F^{\leftarrow}(p)$: inverse of F ("Quantile function").

Using EVT and heavy-machinery from theory of regularly-varying functions:

Max

$$\mathbb{E}[ALG(n)] \approx F^{\leftarrow}\left(1 - \frac{1-\gamma}{n}\right)$$

Min

$$\mathbb{E}[ALG(n)] \approx F^{\leftarrow}\left(\frac{1-\gamma}{n}\right)$$

High-Level Approach

$F(t) = \Pr_{X\sim\mathcal{D}}[X \le t]$, $\quad F^{\leftarrow}(p)$: inverse of F ("Quantile function").

Using EVT and heavy-machinery from theory of regularly-varying functions:

Max

$$\mathbb{E}[ALG(n)] \approx F^{\leftarrow}\left(1 - \frac{1-\gamma}{n}\right)$$

$$\mathbb{E}\left[\max_{i=1}^{n} X_i\right] \approx \Gamma(1-\gamma)\ F^{\leftarrow}\left(1 - \frac{1}{n}\right)$$

Min

$$\mathbb{E}[ALG(n)] \approx F^{\leftarrow}\left(\frac{1-\gamma}{n}\right)$$

$$\mathbb{E}\left[\min_{i=1}^{n} X_i\right] \approx \Gamma(1-\gamma)\ F^{\leftarrow}\left(\frac{1}{n}\right)$$

$$\boxed{\Gamma(x) = (x-1)!}$$

High-Level Approach

$F(t) = \Pr_{X\sim\mathcal{D}}[X \le t]$, $\quad F^{\leftarrow}(p)$: inverse of F ("Quantile function").

Using EVT and heavy-machinery from theory of regularly-varying functions:

<u>**Max**</u>

$$\mathbb{E}[ALG(n)] \approx F^{\leftarrow}\left(1 - \frac{1-\gamma}{n}\right)$$

$$\mathbb{E}\left[\max_{i=1}^{n} X_i\right] \approx \Gamma(1-\gamma)\ F^{\leftarrow}\left(1 - \frac{1}{n}\right)$$

$$F^{\leftarrow}\left(1 - \frac{c}{n}\right) \approx c^{-\gamma}\ F^{\leftarrow}\left(1 - \frac{1}{n}\right)$$

<u>**Min**</u>

$$\mathbb{E}[ALG(n)] \approx F^{\leftarrow}\left(\frac{1-\gamma}{n}\right)$$

$$\mathbb{E}\left[\min_{i=1}^{n} X_i\right] \approx \Gamma(1-\gamma)\ F^{\leftarrow}\left(\frac{1}{n}\right)$$

$$F^{\leftarrow}\left(\frac{c}{n}\right) \approx c^{-\gamma}\ F^{\leftarrow}\left(\frac{1}{n}\right)$$

$$\boxed{\Gamma(x) = (x-1)!}$$

Competition Complexity

Competition Complexity

What if we want $\mathbb{E}[ALG] \geq \mathbb{E}[\max_i X_i]$ or $\mathbb{E}[ALG] \leq \mathbb{E}[\min_i X_i]$?

$\implies$ give ALG more random variables!

What if we want $\mathbb{E}[ALG] \geq \mathbb{E}[\max_i X_i]$ or $\mathbb{E}[ALG] \leq \mathbb{E}[\min_i X_i]$?

$\implies$ give ALG more random variables!

Competition Complexity

For fixed n, the *competition complexity* of a distribution $\mathcal{D}$ is

$$\inf \left\{ c \;\middle|\; \mathbb{E}[ALG(c\, n)] \geq \mathbb{E}[\max_{i=1}^{n} X_i] \right\}$$

for Max

$$\inf \left\{ c \;\middle|\; \mathbb{E}[ALG(c\, n)] \leq \mathbb{E}[\min_{i=1}^{n} X_i] \right\}$$

for Min

- For Max and $\mathcal{D} = \mathcal{D}(n)$, it can be unbounded. [Brustle, Correa, Dütting, Verdugo ’22]

Asymptotic Competition Complexity via EVT

What if we fix $\mathcal{D}$ and take $n \to \infty$?
(Asymptotic Competition Complexity - ACC)

Theorem [L., Verdugo '23]

For every distribution following EVT,

$$ACC_{Max}(\gamma) = ACC_{Min}(\gamma)$$

Asymptotic Competition Complexity via EVT

What if we fix $\mathcal{D}$ and take $n \to \infty$?
(Asymptotic Competition Complexity - ACC)

Theorem [L., Verdugo '23]

For every distribution following EVT,

$$ACC_{Max}(\gamma) = ACC_{Min}(\gamma) = \left(ACR(\gamma)\right)^{-1/\gamma}$$

Asymptotic Competition Complexity via EVT

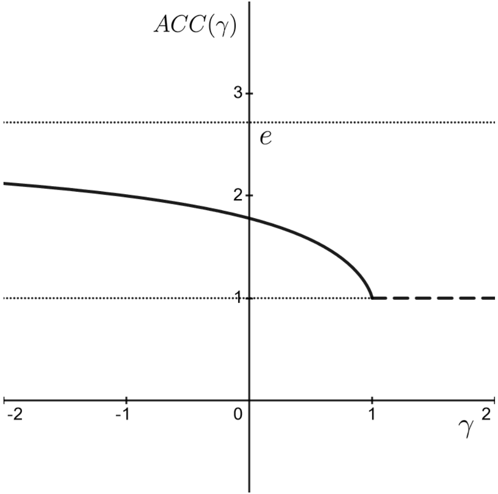
What if we fix $\mathcal{D}$ and take $n \to \infty$?
(Asymptotic Competition Complexity - ACC)

Theorem [L., Verdugo '23]

For every distribution following EVT,

$$ACC_{Max}(\gamma) = ACC_{Min}(\gamma) = \left(ACR(\gamma)\right)^{-1/\gamma} = (1-\gamma)\left(\Gamma(1-\gamma)\right)^{1/\gamma}.$$

Asymptotic Competition Complexity



- $ACC \leq e$ for all $\mathcal{D}$ following EVT.

Overview

1. **Distribution-optimal prophet inequalities** [L., Mehta '22, L. '23]
   - Unified proof for both max and min I.I.D prophet inequality
   - Techniques: Extreme Value Theory, Regularly-Varying Functions
   - Competition complexity
2. **Oracle-augmented prophet inequalities** [Har-Peled, Harb, L. '23]
   - Connection with top-1-of-k model
   - Upper-lower bounds for I.I.D. case
   - Upper-lower bounds for general case (adversarial order)
3. **Optimal greedy OCRSs** [L., '22]
   - $1/e$-selectable greedy OCRS for single-item
   - $1/e$ hardness
   - Extension to transversal matroids
4. **Submodular prophet inequalities** [Chekuri, L. '21]
   - Small constant SPI via OCRS
   - Generalized framework for several constraints
   - Correlation gap

Oracle-Augmented Prophet Inequalities

- $\mathcal{O}_k$ model:
  - Assume ALG has k calls to $\mathcal{O}$, who knows $X_1, \ldots, X_n$.
- Step i:
  - $X_i \geq \max_{j=i+1}^{n} X_j \implies ALG$ selects X_i
  - $X_i < \max_{j=i+1}^{n} X_j \implies ALG$ rejects X_i
- "Algorithms with predictions"

Oracle-Augmented Prophet Inequalities

- $\mathcal{O}_k$ model:
  - Assume ALG has k calls to $\mathcal{O}$, who knows $X_1, \ldots, X_n$.
- Step i:
  - $X_i \geq \max_{j=i+1}^{n} X_j \implies ALG$ selects X_i
  - $X_i < \max_{j=i+1}^{n} X_j \implies ALG$ rejects X_i
- "Algorithms with predictions"

- Top-1-of-k model:
  - ALG selects S with $|S| \leq k$, but only $\max_{X_i \in S} X_i$ matters

[Gilbert, Mosteller '66], [Assaf, Samuel-Cahn '00], [Assaf, Goldstein, Samuel-Cahn '02]

Oracle-Augmented Prophet Inequalities

- $\mathcal{O}_k$ model:
  - Assume ALG has k calls to $\mathcal{O}$, who knows $X_1, \ldots, X_n$.
- Step i:
  - $X_i \geq \max_{j=i+1}^{n} X_j \implies ALG$ selects X_i
  - $X_i < \max_{j=i+1}^{n} X_j \implies ALG$ rejects X_i
- "Algorithms with predictions"

- Top-1-of-k model:
  - ALG selects S with $|S| \leq k$, but only $\max_{X_i \in S} X_i$ matters

[Gilbert, Mosteller '66], [Assaf, Samuel-Cahn '00], [Assaf, Goldstein, Samuel-Cahn '02]

- CR: Maximize $\mathbb{E}[ALG] / \mathbb{E}[\max_i X_i]$
- PbM: Maximize $\Pr[ALG$ selects $\max_i X_i]$

Results for $\mathcal{O}_k$ Model

Theorem [Har-Peled, Harb, L. '23]

- PbM: $\mathcal{O}_k \equiv$ Top-1-of-$(k+1)$

$CR : \frac{\mathbb{E}[ALG]}{\mathbb{E}[\max_i X_i]}$ $\quad PbM : \Pr[ALG \text{ selects } \max_i X_i]$

Results for $\mathcal{O}_k$ Model

Theorem [Har-Peled, Harb, L. '23]

- PbM: $\mathcal{O}_k \equiv$ Top-1-of-$(k+1)$
- CR: $\mathbb{E}\left[ALG\left(\mathcal{O}_k\right)\right] \leq \mathbb{E}\left[OPTALG\left(\text{Top-}1\text{-of-}(k+1)\right)\right]$

$CR : \frac{\mathbb{E}[ALG]}{\mathbb{E}[\max_i X_i]}$ $\qquad$ $PbM : \Pr[ALG \text{ selects } \max_i X_i]$

Results for $\mathcal{O}_k$ Model

Theorem [Har-Peled, Harb, L. '23]

- PbM: $\mathcal{O}_k \equiv$ Top-1-of-$(k+1)$
- CR: $\mathbb{E}\left[ALG\left(\mathcal{O}_k\right)\right] \le \mathbb{E}\left[OPTALG\left(\text{Top-1-of-}(k+1)\right)\right]$
- I.I.D. : $1 - \mathrm{O}\left(k^{-k/5}\right) \le PbM \le 1 - \mathrm{O}\left(k^{-k}\right)$
- General (adversarial order) : $1 - \mathrm{O}\left(e^{-k/5.18}\right) \le CR \le 1 - \mathrm{O}\left(e^{-k/1.44}\right)$

Single-threshold algorithms.

$CR : \frac{\mathbb{E}[ALG]}{\mathbb{E}[\max_i X_i]}$ $\qquad PbM : \Pr[ALG \text{ selects } \max_i X_i]$

Results for $\mathcal{O}_k$ Model

Theorem [Har-Peled, Harb, L. '23]

- PbM: $\mathcal{O}_k \equiv$ Top-1-of-$(k+1)$
- CR: $\mathbb{E}\left[ALG\left(\mathcal{O}_k\right)\right] \leq \mathbb{E}\left[OPTALG\left(\text{Top-1-of-}(k+1)\right)\right]$
- I.I.D. : $1 - \mathrm{O}\left(k^{-k/5}\right) \leq PbM \leq 1 - \mathrm{O}\left(k^{-k}\right)$
- General (adversarial order) :
  $1 - \mathrm{O}\left(e^{-k/5.18}\right) \leq CR \leq 1 - \mathrm{O}\left(e^{-k/1.44}\right)$

Single-threshold algorithms.

$CR : \frac{\mathbb{E}[ALG]}{\mathbb{E}[\max_i X_i]}$ $\quad PbM : \Pr[ALG \text{ selects } \max_i X_i]$

Results for $\mathcal{O}_k$ Model

Theorem [Har-Peled, Harb, L. '23]

- PbM: $\mathcal{O}_k \equiv$ Top-1-of-$(k+1)$
- CR: $\mathbb{E}\left[ALG\left(\mathcal{O}_k\right)\right] \leq \mathbb{E}\left[OPTALG\left(\text{Top-1-of-}(k+1)\right)\right]$
- I.I.D. : $1 - O\left(k^{-k/5}\right) \leq PbM \leq 1 - O\left(k^{-k}\right)$
- General (adversarial order) :
  $1 - O\left(e^{-k/5.18}\right) \leq CR \leq 1 - O\left(e^{-k/1.44}\right)$

Single-threshold algorithms.

Improves upon $1 - O\left(e^{-k/6}\right)$ [Ezra, Feldman, Nehama '18]

$CR : \frac{\mathbb{E}[ALG]}{\mathbb{E}[\max_i X_i]}$ $PbM : \Pr[ALG \text{ selects } \max_i X_i]$

Future Directions

- Principal wants to delegate a PI instance to k agents.
  [Liaw, L., Perlroth, Schvartzman, Wang '24].
- Beyond the independence assumption.
  [L., Patton, Singla '24].
- Multiple selection minimization PI.
- Free-Order PI: ALG can choose order of realizations.
  Can we get ≈ 0.745 (I.I.D. constant)?

♥ Thank You ♥





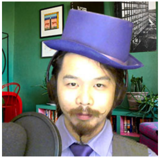



♥ Thank You ♥

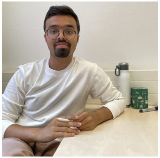


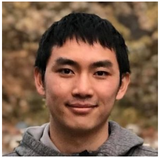




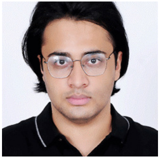



♥ Thank You ♥

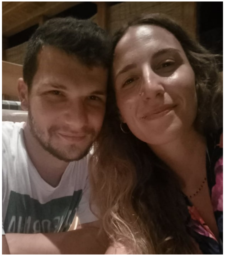


♥ Thank You ♥

♥ Thank You ♥

♥ Thank You ♥



Questions?