Minimization is Harder in the Prophet World

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Joint work with

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Prophet Inequality

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- Buyer i has private valuation v_i. How to offer prices?
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Prophet Inequality [Krengel, Sucheston, Garling '77]

$X_1, X_2, \ldots, X_n \sim$ (known) $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$
arrive in *adversarial* order.

- Design *stopping time* to maximize selected value.
- Step i: Take-it-or-leave-it decision.
- Compare against all-knowing *prophet*: $\mathbb{E}[\max_i X_i]$.

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$\mathcal{U}[7, 16]$

$\mathcal{U}[0, 20]$

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$X_1 = 13.93$	$X_2 = 8.15$	$X_3 = 5.60$	$X_4 = 0$

$$\mathbb{E}[\max\{X_1, X_2, X_3, X_4\}] \approx 24.66$$

$$\mathbb{E}[OPTALG\{X_1, X_2, X_3, X_4\}] \approx 13.37$$

Optimal strategy was to continue through to X_4.

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$\mathbb{E}\,[\text{ALG}] = 1$ for all algorithms.

$\mathbb{E}[\max\{X_1, X_2\}] = \frac{1}{\varepsilon} \cdot \varepsilon + 1 \cdot (1-\varepsilon) = 2 - \varepsilon$.

- Applications in Posted Price Mechanisms.
  [Hajiaghayi, Kleinberg, Sandholm '07, Chawla, Hartline, Malec, Sivan '10]

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  $T = \mathbb{E}[\max_i X_i]/2$ [Kleinberg, Weinberg '12]

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[Hill, Kertz '82]
For any $\mathcal{D}$, $\exists$ a single threshold τ such that selecting the first $X_i \geq \tau$ achieves $(1 - 1/e) \cdot \mathbb{E}[\max_i X_i] \approx 0.632 \cdot \mathbb{E}[\max_i X_i]$.

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[Correa, Foncea, Hoeksma, Oosterwijk, Vredeveld '21]

For any $\mathcal{D}$, $\exists$ threshold stopping strategy $\tau_1, \tau_2, \ldots, \tau_n$ that achieves $\beta \cdot \mathbb{E}[\max_i X_i]$, where $\beta \approx 0.745$.

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What if objective is $\min_i X_i$? Same problem?

- Objective: Minimize selected value, compare against $\overline{\mathbb{E}[\min_i X_i]}$.
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- No bound on competitive ratio for general distributions. [Esfandiari, Hajiaghayi, Liaghat, Monemizadeh '15]

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Intuition:
Set $T = c \cdot \mathbb{E}[\min_i X_i]$.

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No hope for universal bound: [Lucier '22]

$\mathcal{D} : F(x) = 1 - {}^1\!/_x$, with $x \in [1, +\infty)$ (Equal-revenue distribution).
$\mathbb{E}[X] = 1 + \int_1^\infty (1 - F(x))\, dx = +\infty$, but

$\mathbb{E}[\min\{X_1, X_2\}] = 1 + \int_1^\infty (1 - F(x))^2\, dx = 2 < +\infty$.

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Idea

Look at "fatness" of $\mathcal{D}$'s tail. Captured by $\mathcal{D}$'s *Hazard Rate*.

$$h(x) = \frac{f(x)}{1 - F(x)} \qquad \left(\text{also } H(x) = \int_0^x h(u)\,du\right)$$

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h is increasing.

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Entire Distribution

$\mathcal{D}$ is *entire* if H has convergent series expansion $H(x) = \sum_{i=1}^{\infty} a_i x^{d_i}$ (where $0 < d_1 < d_2 < \ldots$) for every x in the support of $\mathcal{D}$.

- E.g. uniform, exponential, Gaussian, Weibull, Rayleigh, beta

Single Threshold

Single threshold algorithms suffice for maximization.

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Theorem 1

For every entire distribution $\mathcal{D}$, $\exists$ a single threshold algorithm that is $\Theta\left((\log n)^{1/d_1}\right)$-competitive, and this is the best possible.

- Cannot achieve a constant approximation, even for "easy" distributions with very light tail, e.g. exponential.
- d_1: Valuation of H, only important quantity!

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Optimal policy: Set $\tau_i = \mathbb{E}[\mathrm{OPTALG}_{i+1,\ldots,n}]$, accept first $X_i \leq \tau_i$.

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For every distribution,

- If $\mathbb{E}[X] = +\infty$, the competitive ratio is infinite.
- If $\mathbb{E}[X] < +\infty$, there exists a constant c-competitive minimization prophet inequality, and we characterize the *optimal* c as the solution to a simple inequality on the quantiles of $\mathcal{D}$.

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- c is distribution-dependent – can be arbitrarily large.
- First distribution-sensitive guarantees for prophet inequalities.
- New use of hazard rate in prophet inequalities as *analysis tool*.

More Positive Results

Theorem 3

For every MHR distribution, there exists a 2-competitive minimization prophet inequality, and this is the best-possible.

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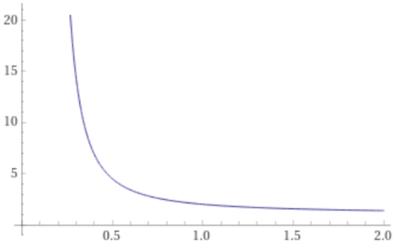
What does c look like?

Theorem 4

For every *entire* distribution,

$$c(d_1) = \frac{(1 + {}^{1}/_{d_1})^{1/d_1}}{\Gamma\left(1 + {}^{1}/_{d_1}\right)} = \Theta\left(e^{1/d_1}\right).$$



- Γ: Gamma function. $\Gamma(n+1) = n!$

Open Problems

- Extend minimization PI to multiple selection.
- What can you get with $1 < k < n$ thresholds?
- Distribution-sensitive constants for maximization?

Questions?

Good Question!

Hidden under the rag:

$$c = \inf \left\{ c' \;\middle|\; c' - \frac{F^{-1}\left(\frac{\lambda_{\mathcal{D}}\left(1+\frac{1}{c'-1}\right)}{n}\right)}{F^{-1}\left(\frac{1}{n}\right)} \geq 0 \right\}.$$

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$$T = \mathrm{O}\left(\left(\frac{\log n}{n}\right)^{1/d_1}\right).$$