Matroid Secretary via Labeling Schemes

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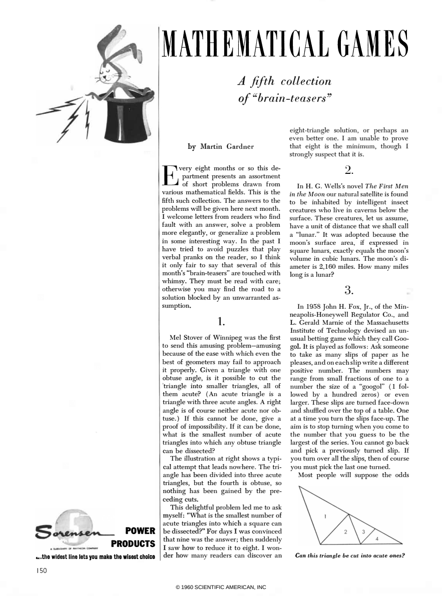

MATHEMATICAL GAMES

A fifth collection of "brain-teasers"

by Martin Gardner

Every eight months or so this department presents an assortment of short problems drawn from various mathematical fields. This is the fifth such collection. The answers to the problems will be given here next month. I welcome letters from readers who find fault with an answer, solve a problem more elegantly, or generalize a problem in some interesting way. In the past I have tried to avoid puzzles that play verbal pranks on the reader, so I think it only fair to say that several of this month's "brain-teasers" are touched with whimsy. They must be read with care; otherwise you may find the road to a solution blocked by an unwarranted assumption.

1.

Mel Stover of Winnipeg was the first to send this amusing problem—amusing because of the ease with which even the best of geometers may fail to approach it properly. Given a triangle with one obtuse angle, is it possible to cut the triangle into smaller triangles, all of them acute? (An acute triangle is a triangle with three acute angles. A right angle is of course neither acute nor obtuse.) If this cannot be done, give a proof of impossibility. If it can be done, what is the smallest number of acute triangles into which any obtuse triangle can be dissected?

The illustration at right shows a typical attempt that leads nowhere. The triangle has been divided into three acute triangles, but the fourth is obtuse, so nothing has been gained by the preceding cuts.

This delightful problem led me to ask myself: "What is the smallest number of acute triangles into which a square can be dissected?" For days I was convinced that nine was the answer; then suddenly I saw how to reduce it to eight. I wonder how many readers can discover an eight-triangle solution, or perhaps an even better one. I am unable to prove that eight is the minimum, though I strongly suspect that it is.

2.

In H. G. Wells's novel *The First Men in the Moon* our natural satellite is found to be inhabited by intelligent insect creatures who live in caverns below the surface. These creatures, let us assume, have a unit of distance that we shall call a "lunar." It was adopted because the moon's surface area, if expressed in square lunars, exactly equals the moon's volume in cubic lunars. The moon's diameter is 2,160 miles. How many miles long is a lunar?

3.

In 1958 John H. Fox, Jr., of the Minneapolis-Honeywell Regulator Co., and L. Gerald Marnie of the Massachusetts Institute of Technology devised an unusual betting game which they call Googol. It is played as follows: Ask someone to take as many slips of paper as he pleases, and on each slip write a different positive number. The numbers may range from small fractions of one to a number the size of a "googol" (1 followed by a hundred zeros) or even larger. These slips are turned face-down and shuffled over the top of a table. One at a time you turn the slips face-up. The aim is to stop turning when you come to the number that you guess to be the largest of the series. You cannot go back and pick a previously turned slip. If you turn over all the slips, then of course you must pick the last one turned.

Most people will suppose the odds

Can this triangle be cut into acute ones?

Secretary Problem

- n *unknown* values $w_1, \dots, w_n$
- Random order
- Step i:
  1. Select w_i and stop
  2. Ignore w_i and continue

$\Pr[\text{We select } \max_i w_i]$?

Secretary Problem

S_1
Sampling Phase

S_2
Selection Phase

①

S_1
Sampling Phase

S_2
Selection Phase

Secretary Problem

① ④ …

S_1
Sampling Phase

S_2
Selection Phase

1 4 … −2

S_1
Sampling Phase

S_2
Selection Phase

Secretary Problem

1 4 … −2 0

S_1
Sampling Phase

S_2
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Secretary Problem

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S_1
Sampling Phase

0 7 …

S_2
Selection Phase

Secretary Problem

1 4 … −2 0 7 … 11

S_1
Sampling Phase

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$$\left.\begin{array}{ll}\text{w.p. } {}^1\!/_2, & w_1^* \in S_2 \\ \text{w.p. } {}^1\!/_2, & w_2^* \in S_1\end{array}\right\} \implies Pr[\text{We select} \max_i w_i] \geq {}^1\!/_4$$

Secretary Problem

① ④ … ⊖2   ⓪ ⑦ … ⑪

S_1
Sampling Phase

S_2
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$$\left.\begin{array}{ll}\text{w.p. } 1/2, & w_1^* \in S_2 \\ \text{w.p. } 1/2, & w_2^* \in S_1\end{array}\right\} \implies Pr[\text{We select} \max_i w_i] \geq 1/4$$

- Optimal: fix arrival time t of w_1^*.

$$Pr[ALG \leftarrow w_1^*] = \int_p^1 \Pr\left[\text{Largest element before } t \in S_1\right] dt$$

$$= \int_p^1 \frac{p}{t}\, dt = p \ln\left(\frac{1}{p}\right) = 1/e \text{ for } p = 1/e.$$

Generalizations?

Given constraints $\mathcal{F}$ and (unknown) weights w on elements E, select $S \subseteq E$

- *online* in uniformly random order,
- $S \in \mathcal{F}$ (*feasible*),
- to *maximize* $w(S) = \sum_{e \in S} w_e$

Compare against $OPT = \max_{T \in \mathcal{F}} w(T)$

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3. Matroids

Matroid Secretary

Matroid Secretary Conjecture [BIK '07]

Given matroid $M = (E, \mathcal{F})$, observe weight w of elements of E in a uniformly random order. Then, $\exists$ $c > 0$ and algorithm $\mathcal{A}$ which selects $S \subseteq E$ immediately and irrevocably s.t.

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Strong Matroid Secretary Conjecture [BIK '07]

The Matroid Secretary Conjecture holds for $c = 1/e$ for all matroids.

Simplest Matroid?

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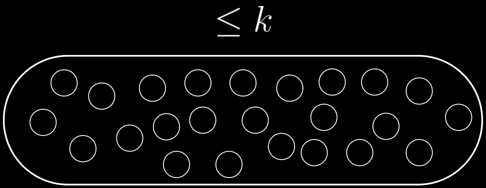
k-Uniform Matroid



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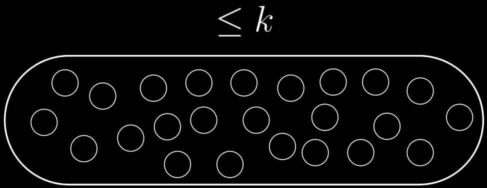
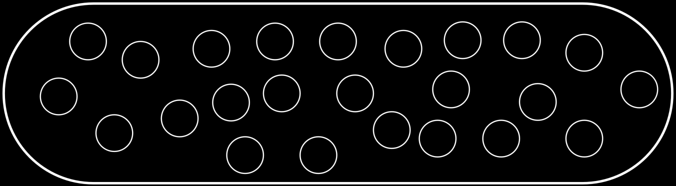


Can get $\left(1 - O\left(1/\sqrt{k}\right)\right)$-approx. to OPT [K '05]

Slightly more complicated?

$\leq k_1$



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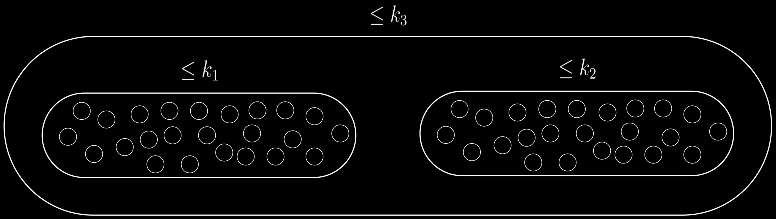

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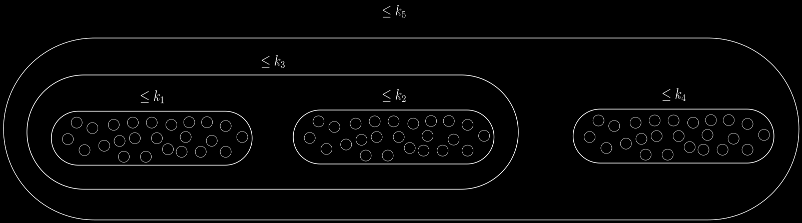

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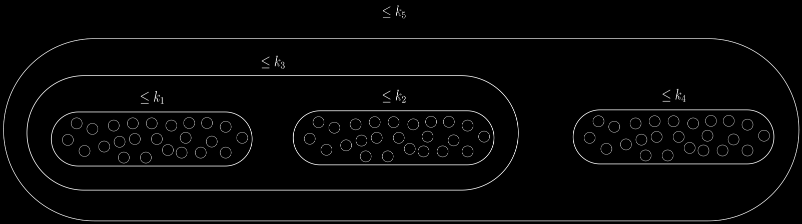
Laminar Matroid



Fix a "sampling" parameter p.

Greedy Improving Algorithm (p)

- $S \leftarrow \emptyset$
- For $i \leftarrow 1$ to $\lceil p\, n \rceil$
  - Skip i
- For $i \leftarrow \lceil p\, n \rceil + 1$ to n
  - Observe w_i
  - If $S + i \in \mathcal{F}$ and $i \in OPT_{\leq i}$
    - $S \leftarrow S + i$
- Return S

State-of-the-Art

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- Greedy Improving Algorithm

Our Contributions

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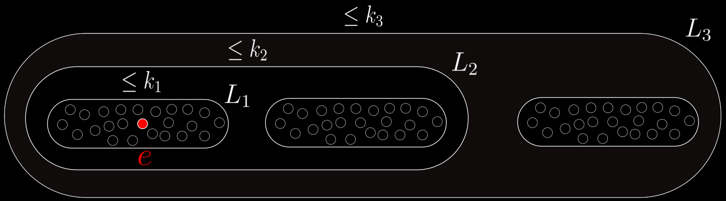
Key Issue



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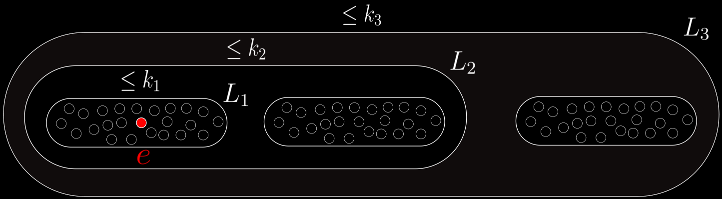

- Want to calculate

$$\Pr[\,\exists \text{ space for } e\,] =$$
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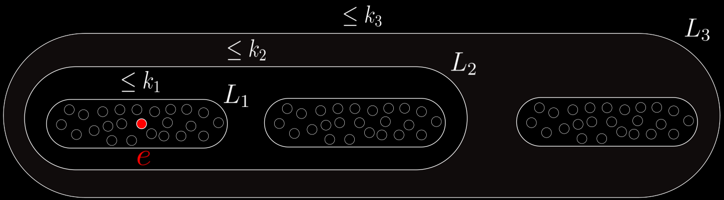

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- Computing $\Pr[\,|S \cap L_i| \leq k_i - 1\,]$ is easy but

$$|S \cap L_i| \leq k_i - 1 \quad \text{and} \quad |S \cap L_j| \leq k_j - 1$$

are *correlated* events!

Tools and Techniques

- Previous approaches (IW'11, MTW'13, HPZ'24):
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- Our approach: tight correlation analysis via labeling scheme

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1. We show $N[a,b) \sim Poi\left(r \cdot \ln(b/a)\right)$
   - $S(t)$: last *improving* element in $[0,t)$

$$\Pr[S(b) \le x] = \prod_{e \in OPT(E_x)} \Pr[t_e \le x] = \left(\frac{x}{b}\right)^r.$$

   - $y_0 = 1, y_k = S(y_{k-1})$. Also, $x_k \triangleq -\ln(y_k)$.

$$\begin{aligned}\Pr[x_k - x_{k-1} \le x] &= \Pr\left[\ln(y_{k-1}) - \ln(y_k) \le x\right] \\ &= \Pr\left[S(y_{k-1}) \ge y_{k-1}e^{-x}\right] = 1 - \left(\frac{y_{k-1}e^{-x}}{y_{k-1}}\right)^r \\ &= 1 - e^{-xr}.\end{aligned}$$

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At each *improving element* e assign a label $\ell(e)$ equal to its *relative rank* at the time of arrival.

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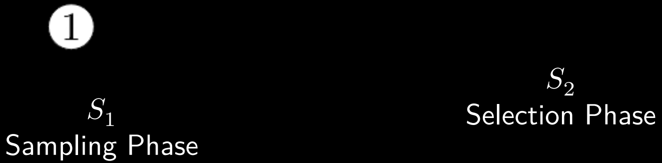

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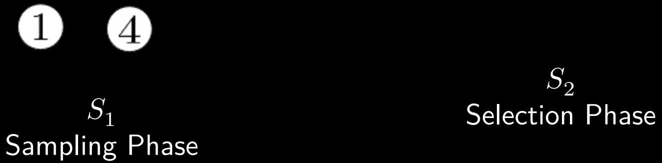

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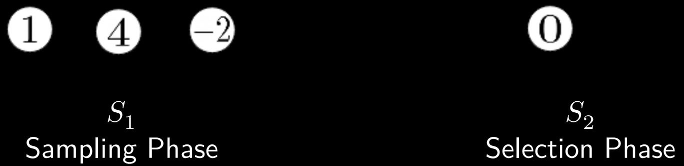

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(1) (4) (−2) (0) (7)

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$1 \quad 4 \quad -2$

S_1
Sampling Phase

$0 \quad 7 \quad 11$

S_2
Selection Phase

e_4 is not improving, $\quad \ell(e_5) = 1, \quad \ell(e_6) = 1$

When is e Selected?

- Fix $e \in OPT$. e is selected iff

$$|S \cap L_1| \leq k_1 - 1 \;\; \wedge \;\; |S \cap L_2| \leq k_2 - 1 \;\; \wedge \;\; \ldots$$

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Let $y^{(e)}$ denote the labels of improving elements *before* e

$\Longrightarrow$ suffices that, for every chain set $L_j \ni e$ with $rank(L_j) = k_j$

$$|\left\{ i \in [|y^{(e)}|] \;\middle|\; y_i^{(e)} \leq k_j \right\}| \leq k_j - 1.$$

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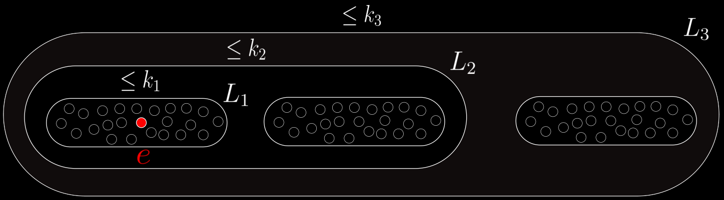
To see this, order $y^{(e)}$ from "inner" to "outer" chains.



Language of a Matroid

$\forall M$, create $\mathcal{L}_M$ s.t. $\forall e \in OPT$

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- Graphic:

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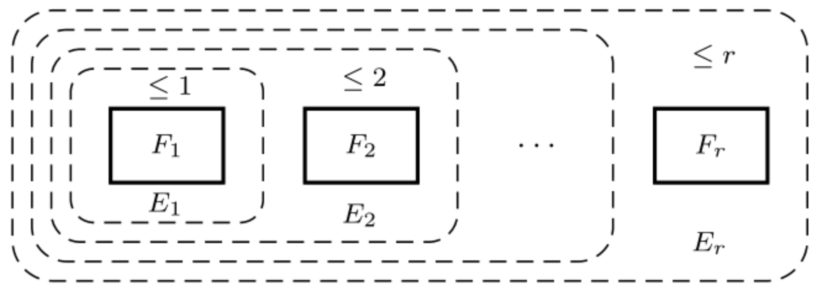
- Graphic: ...even more complicated!

$\implies \Pr\left[y^{(e)} \in \mathcal{L}_M\right] \geq 2 - 2p + \ln(p)$, maximized for $p = 1/2 \implies 1 - \ln(2) \approx 0.3068$-approximation.

Result for Laminar Matroids

$\implies \Pr\left[y^{(e)} \in \mathcal{L}_M\right] \geq 2 - 2p + \ln(p)$, maximized for $p = 1/2 \implies 1 - \ln(2) \approx 0.3068$-approximation.

Tight for



Beyond Greedy Improving

- Labeling schemes also work for graphic matroids!

  Clever scheme (thanks José) avoids cycles without dropping too many edges

  $\implies 0.2504$-approximation.

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- Can beat Greedy Improving for rank-2 matroids...yet $^1/_e$ is still open...

Conclusion

- Technique also subsumes prior work on special classes of matroids.
- Hopefully can be used on
  - matroid classes for which the conjecture is still open (e.g. gammoids), to give constant-factor algorithms.
  - matroid classes for which a constant is known to give a $1/e$-approximation.

Questions?