(Almost) Envy-Free, Proportional and Efficient Allocations of an Indivisible Mixed Manna

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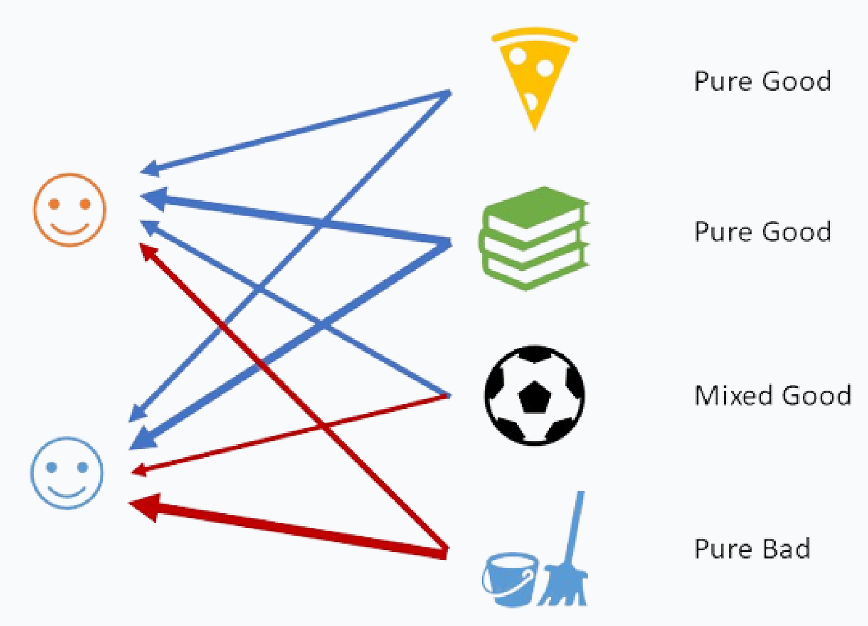
Model

- n agents: N.
- m **indivisible** items: M.
- Valuations $v_i(S)$, $\forall$ agents i and sets of items S.

- **Goal:** Find allocation A that is
  - **Fair:** EFX and PropMX,
  - **Efficient:** Pareto-Optimality (PO).

- **Challenge:** Items are **mixed manna**, i.e. item j might be a **good** for some agent i ($v_{ij} \geq 0$) and a **bad** for another agent i' ($v_{i'j} < 0$).

- Types of **Instances:**
  - **Separable:** $M = M^+ \cup M^-$, where M^+ are goods for all and M^- are bads for all.
  - **Restricted Mixed Goods (RMG):** $\forall j \in M$, either $v_{ij} \leq 0$ or $v_{ij} = v_j$ for all agents i.



Line thickness indicates the magnitude of absolute value.

- Model **generalizes** standard fair division of goods or bads.

Results

- All allocations are **efficiently computable**.
- **Separable** instances: **PropMX** allocation.
- **RMG** instances:
  - **PropMX** allocation for **general pure bads**.
  - **EFX + PropMX** allocation for **IDO pure bads**.
  - **EFX + PropMX + PO** allocation for **identical pure bads**.

Background

- An allocation A is:
  - **EFX:** $\forall i, i' \in N$, either we have that $\forall j \in A_{i'}$ where $v_{ij} > 0$,
$$v_i(A_i) \geq v_i(A_{i'}) - v_{ij},$$
or we have that $\forall j \in A_i$ where $v_{ij} < 0$,
$$v_i(A_i) - v_{ij} \geq v_i(A_{i'}).$$
  - **PropMX:** $\forall i \in N$, either we have
$$v_i(A_i) + \max_{i' \neq i} \min_{j \in A_{i'} : v_{ij} > 0} v_{ij} \geq \frac{1}{n} \cdot v_i(M),$$
or $\forall j \in A_i$ where $v_{ij} < 0$, we have
$$v_i(A_i) - v_{ij} \geq \frac{1}{n} \cdot v_i(M).$$
  - **PO:** If $\nexists$ allocation A' such that $v_i(A'_i) \geq v_i(A_i)$ for all $i \in N$ and $v_i(A'_i) > v_i(A_i)$ for at least one $i \in N$.

- **Pure bads:** Set M^- of items such that $v_{ij} < 0$ for all $i \in N$ and $j \in M^-$.
  - M^- is **IDO** if: $\exists$ ordering of M^- such that $\forall i \in N, v_{i1} \leq v_{i2} \leq \cdots \leq v_{i|M^-|}$.
  - M^- is **identical** if: $\forall j \in M^-$, $\exists v_j < 0$ such that $\forall i \in N$, $v_{ij} = v_j$.

Approach

- We develop algorithm RestrictedGoods which finds an **EFX + PO allocation** for RMG instances with **goods only**, by modifying the envy-cycle elimination algorithm.
- The following algorithm can be modified for **RMG instances** with **IDO bads** to allocate the bads in IDO order and return an **EFX** allocation.
- The allocation of M^- can be further modified for **RMG instances** with **general bads** to return a **PropMX** allocation.
- Our results **generalize** previous work on binary and identical mixed manna [Aleksandrov and Walsh '19, '20].

Algorithm

Algorithm: EFX + PO for RMG instances with identical bads

$M \to M^+ \uplus M^0 \uplus M^-$, where

- $j \in M^+ \implies \exists i \in N : v_{ij} > 0$,
- $j \in M^0 \implies \exists i \in N : v_{ij} = 0$,
- $j \in M^- \implies \forall i \in N : v_{ij} < 0$.

Phase 1: Allocate all items $j \in M^+$ under modified valuations v' using algorithm RestrictedGoods:
$$v'_{ij} = \begin{cases} v_{ij} & \text{if } v_{ij} \geq 0 \\ 0 & \text{if } v_{ij} < 0 \end{cases}.$$

Phase 2: Allocate all items $j \in M^0$ to agents i such that $v_{ij} = 0$.

Phase 3: Allocate all items $j \in M^-$ in decreasing order of disutility to an agent that is a **sink** in the **envy-graph** of the current allocation.

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