A Unified and Distribution-Optimal Analysis of the Max and Min IID Prophet Inequality

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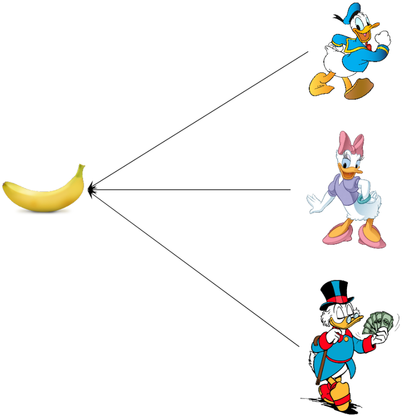

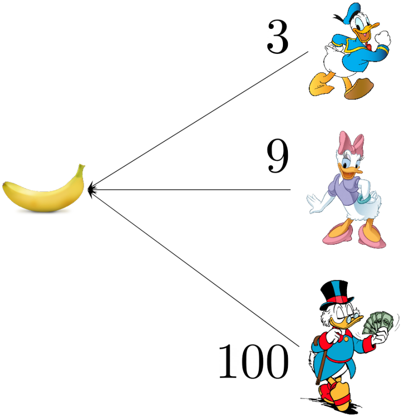
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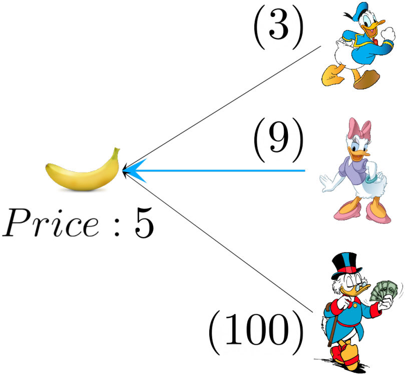
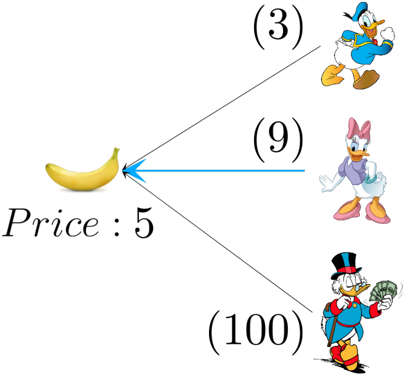


(3)
(9)
Price : 5
(100)



How to set the price?

Prophet Inequality

[Krengel, Sucheston and Garling '77]

$X_1, X_2, \dots, X_n \sim$ (known) $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n$
arrive in *adversarial* order.

- Design *stopping time* to maximize selected value.
- Compare against all-knowing *prophet*: $\mathbb{E}[\max_i X_i]$.
- Competitive Ratio:

$$\frac{\mathbb{E}[ALG]}{\mathbb{E}[\max_i X_i]}.$$

$\mathcal{U}[2,4]$

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$X_1 = 2.34$

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Prophet Inequality [Krengel, Sucheston and Garling '77, '78]

$\exists$ stopping strategy that achieves $1/2 \cdot \mathbb{E}[\max_i X_i]$,
and this is tight.

$$X_1 = 1 \quad \text{w.p. } 1, \text{ and } X_2 = \begin{cases} 1/\varepsilon & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1-\varepsilon \end{cases}$$

$\mathbb{E}[\mathsf{ALG}] = 1$ for all algorithms.

$\mathbb{E}[\max_i X_i] = \frac{1}{\varepsilon} \cdot \varepsilon + 1 \cdot (1-\varepsilon) = 2 - \varepsilon$.

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- Idea: Set *threshold* T, accept first $X_i \geq T$.
  - $T : \Pr[\max_i X_i \geq T] = 1/2$ works [Samuel-Cahn '84].
  - $T = 1/2 \cdot \mathbb{E}[\max_i X_i]$ works [Wittmann '95, Kleinberg and Weinberg '12].

1. When can we get better guarantees?
   Can $1/2$ be improved?
2. What if $\mathcal{D}_1 = \mathcal{D}_2 = \cdots = \mathcal{D}_n = \mathcal{D}$?

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4. Worst-case $\mathcal{D}$?

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IID Prophet Inequality [Hill-Kertz '82, Correa, Foncea, Hoeksma, Oosterwijk and Vredeveld '21]]

For any $\mathcal{D}$, $\exists$ threshold stopping strategy $\tau_1, \tau_2, \dots, \tau_n$ that achieves $\beta \cdot \mathbb{E}[\max_i X_i]$, where $\beta \approx 0.745$, and this is tight.

Worst-case $\mathcal{D}$: High variance – depends on n
Most of the mass is at 0 – low probability of getting a high value.

Optimal Threshold DP

If we reach X_n, take it. Focus on X_{n-1}, X_n.
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$$\mathbb{E}[\mathrm{OPTALG}_{n-1,n}] = (1 - F(\tau_{n-1}))\,\mathbb{E}\left[X \mid X \geq \tau_{n-1}\right] + F(\tau_{n-1})\,\mathbb{E}[X]$$
$$\implies \tau_{n-1} = \mathbb{E}[X].$$

In general, we have $\tau_i = \mathbb{E}[\mathrm{OPTALG}_{i+1,\ldots,n}]$.

Beyond Constant Factors?

Idea

Look at "fatness" of $\mathcal{D}$'s tail. Captured by $\mathcal{D}$'s *Hazard Rate*

$$h(x) = \frac{f(x)}{1 - F(x)}.$$

Intuition: $h(x) = \Pr\left[X = x \mid X \geq x\right]$ (for discrete distributions).

MHR Distribution

h is increasing.

- Important subclass, lots of past work by economists. Good guarantees in many applications (e.g. revenue maximization in auctions).

IID Prophet Inequality with MHR Distribution [Braun-Buttkus-Kesselheim '21]

For any MHR $\mathcal{D}$, $\exists$ threshold algorithm ALG that achieves $\frac{\mathbb{E}[ALG]}{\mathbb{E}[\max_i X_i]} = 1 - \mathrm{o}\left(1\right).$

Minimization

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1. No hope for non-IID:

$$X_1 = 1 \quad \text{w.p. } 1, \text{ and } X_2 = \begin{cases} 1/\varepsilon & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1-\varepsilon \end{cases}$$

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2. No hope for universal bound: [Lucier '22]

$\mathcal{D} : F(x) = 1 - 1/x$, with $x \in [1, +\infty)$ (Equal-revenue distribution).
$\mathbb{E}[X] = 1 + \int_1^\infty (1 - F(x))\, dx = +\infty$, but

$\mathbb{E}[\min\{X_1, X_2\}] = 1 + \int_1^\infty (1 - F(x))^2\, dx < +\infty$.

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Asymptotic Competitive Ratio (ACR)

$$\lambda_{max} = \lim_{n\to\infty} \frac{\mathbb{E}[ALG(n)]}{\mathbb{E}[\max_{i=1}^{n} X_i]} \qquad \lambda_{min} = \lim_{n\to\infty} \frac{\mathbb{E}[ALG(n)]}{\mathbb{E}[\min_{i=1}^{n} X_i]}$$

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- Distribution of M_n, m_n as $n \to \infty$?
- $\lim_{n\to\infty} M_n = +\infty, \quad \lim_{n\to\infty} m_n = 0 \implies$ Re-scaling

Main Tool: Extreme Value Theory

Extreme Value Theorem [Fisher, Tippett '28, Gnedenko '43]

Assume there exist sequences $a_n > 0, b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} F_{M_n}(a_n x + b_n) = G_\gamma^+(x).$$

Then,

$$G_\gamma^+(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), & \text{if } \gamma \neq 0 \\ \exp\left(-\exp\left(-x\right)\right), & \text{if } \gamma = 0 \end{cases},$$

and we say that $\mathcal{D}$ follows EVT.

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- G : Extreme Value Distribution, γ : Extreme Value Index
- Three distinct G_γ^+'s:
  - $\gamma > 0$: Fréchet
  - $\gamma = 0$: Gumbel
  - $\gamma < 0$: Reverse Weibull

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- Can get similar result for discrete distributions.

Fréchet, Gumbel and Rev. Weibull

$$G_\gamma^+(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), & \text{if } \gamma \neq 0 \\ \exp\left(-\exp\left(-x\right)\right), & \text{if } \gamma = 0 \end{cases}$$

- $\gamma > 0$: Fréchet

$$\lim_{x\to\infty} 1 - G_\gamma^+(x) \sim \frac{1}{(\gamma x)^{-1/\gamma}}.$$

Heavy tails, $\nexists$ moments of order $1/\gamma$ and above.
Examples: Cauchy, Pareto, Equal-Revenue, ...

- $\gamma = 0$: Gumbel

$$\lim_{x\to\infty} 1 - G_0^+(x) \sim e^{-x}.$$

Light tail, exponential-like behaviour.
Examples: Gaussian, Exponential, Gamma, ...

- $\gamma < 0$: Reverse Weibull
Necessarily bounded support, short tail.
Examples: Uniform, Beta, ...

Comparison with CLT

Central Limit Theorem

For a $\mathcal{D}$ with mean μ and variance σ^2, let $a_n = \sqrt{n}$ and $b_n = n\mu$. If $\mu, \sigma^2 < +\infty$, then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i - b_n}{a_n} = Y \sim \mathcal{N}\left(0, \sigma^2\right).$$

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Extreme Value Theorem

For a $\mathcal{D}$ with cdf F, let $b_n = (1/(1-F))^{\leftarrow}(n)$, $a_n = 1/(nf(b_n))$. If

$$\lim_{n\to\infty} \frac{\max\{X_1, \ldots, X_n\} - b_n}{a_n} = Y$$

exists, then $Y \sim G_\gamma^+(x)$.

IID PI via Extreme Value Theory

$$\boxed{\Gamma(x) = (x-1)!}$$

Theorem

Assume there exist sequences $a_n > 0, b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} F_{M_n}(a_n x + b_n) = G_\gamma^+(x)$$

for some γ

$$\lim_{n\to\infty} F_{m_n}(a_n x + b_n) = G_\gamma^-(x)$$

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Then, the optimal DP achieves a competitive ratio, as $n \to \infty$, of

$$ACR_{Max} = \min\left\{\frac{(1-\gamma)^{-\gamma}}{\Gamma(1-\gamma)}, 1\right\}.$$

[Kennedy, Kertz '91]

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[L. Mehta '24]

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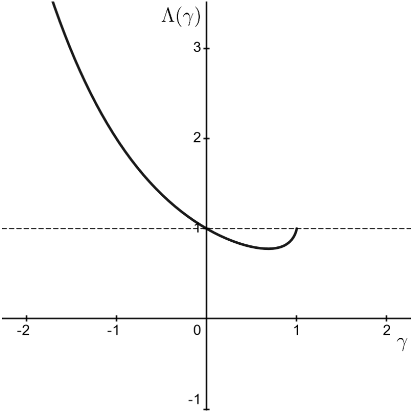
[L. Mehta '24]

- Distribution-optimal closed form!
- Unified analysis of competitive ratio for both $\textsc{Max}$ and $\textsc{Min}$.
- $\mathcal{D}$ MHR $\implies$ $ACR_{Max} = 1 \quad \& \quad ACR_{Min} \leq 2$

Asymptotic Competitive Ratio



For $\gamma \to -\infty$, by Stirling's approximation

$$\frac{(1-\gamma)^{-\gamma}}{\Gamma(1-\gamma)} \approx e^{-\gamma}.$$

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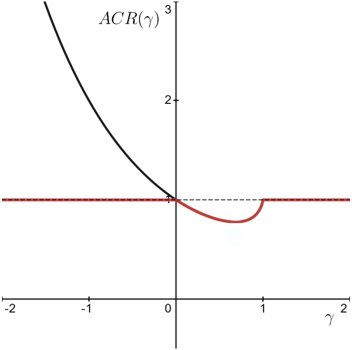

Figure: ACR(γ) for MAX

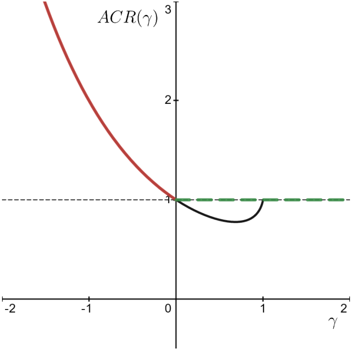

Figure: ACR(γ) for MIN

High-Level Approach

$F(t) = \Pr_{X \sim \mathcal{D}}[X \leq t]$, $\quad F^{\leftarrow}(p)$: inverse of F ("Quantile function").

Using EVT and heavy-machinery from theory of regularly-varying functions:

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Single-Threshold Algorithms?

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Theorem

Assume there exist sequences $a_n > 0, b_n \in \mathbb{R}$ such that

$\lim_{n\to\infty} F_{M_n}(a_n x + b_n) = G^+_\gamma(x)$	$\lim_{n\to\infty} F_{m_n}(a_n x + b_n) = G^-_\gamma(x)$
for some γ	for some γ

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Then, the optimal single-threshold algorithm achieves a competitive ratio, as $n \to \infty$, of

$$ACR_{Max}(T) = \min\left\{ \frac{-\left(-\gamma - W_{-1}\left(-\gamma\, e^{-\gamma}\right)\right)^{1-\gamma}}{(1-\gamma)\Gamma(1-\gamma)W_{-1}\left(-\gamma\, e^{-\gamma}\right)}, \right.$$

$$1\ .$$

[Correa, Pizarro, Verdugo '21]]

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[Correa, Pizarro, Verdugo '21]]

$$ACR_{Min}(T) = \mathrm{O}\left((\log n)^{-\gamma}\right)$$

[L. Mehta '24]

Asymptotic Competition Complexity

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Asymptotic Competition Complexity (ACC)

For any $\mathcal{D}$,

$$ACC_{Max} = \lim_{n\to\infty} \inf\left\{c \,\middle|\, \mathbb{E}[ALG(c\,n)] \geq \mathbb{E}[\max_{i=1}^{n} X_i]\right\}$$

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Theorem [L. '24]

For every distribution following EVT,

$$ACC_{Max}(\gamma) = ACC_{Min}(\gamma)$$

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$$ACC_{Min} = \lim_{n\to\infty} \inf\left\{ c \,\middle|\, \mathbb{E}[ALG(c\,n)] \leq \mathbb{E}[\min_{i=1}^{n} X_i] \right\}$$

Theorem [L. '24]

For every distribution following EVT,

$$ACC_{Max}(\gamma) = ACC_{Min}(\gamma) = \Big(ACR(\gamma)\Big)^{-1/\gamma}$$

Asymptotic Competition Complexity

Asymptotic Competition Complexity (ACC)

For any $\mathcal{D}$,

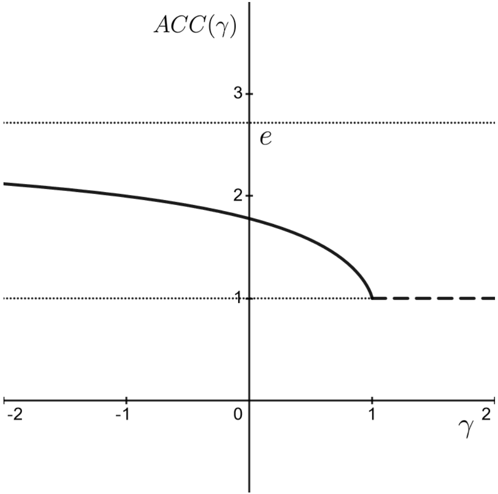
$$ACC_{Max} = \lim_{n\to\infty} \inf\left\{c \,\middle|\, \mathbb{E}[ALG(c\,n)] \geq \mathbb{E}[\max_{i=1}^{n} X_i]\right\}$$
$$ACC_{Min} = \lim_{n\to\infty} \inf\left\{c \,\middle|\, \mathbb{E}[ALG(c\,n)] \leq \mathbb{E}[\min_{i=1}^{n} X_i]\right\}$$

Theorem [L. '24]

For every distribution following EVT,

$$ACC_{Max}(\gamma) = ACC_{Min}(\gamma) = \Big(ACR(\gamma)\Big)^{-1/\gamma} = (1-\gamma)\left(\Gamma(1-\gamma)\right)^{1/\gamma}.$$

Asymptotic Competition Complexity



- $ACC \leq e$ for all $\mathcal{D}$ following EVT.

Open Problems

- Extend Min-PI to multiple selection.
- Are there $\mathcal{D}_i$ for which we can get constant approximation in the non-IID setting?
- What can you get with $1 < k < n$ thresholds?

Questions?