Matroid Secretary via Labeling Schemes

Vasilis Livanos

Center for Mathemathical Modeling (CMM), Santiago, Chile

Joint work with

Kristóf Bérczi
ELTE, Hungary

Jose Soto
U. of Chile

Victor Verdugo
PUC, Chile

MATHEMATICAL GAMES

A fifth collection of "brain-teasers"

by Martin Gardner

Every eight months or so this department presents an assortment of short problems drawn from various mathematical fields. This is the fifth such collection. The answers to the problems will be given here next month. I welcome letters from readers who find fault with an answer, solve a problem more elegantly, or generalize a problem in some interesting way. In the past I have tried to avoid puzzles that play verbal pranks on the reader, so I think it only fair to say that several of this month's "brain-teasers" are touched with whimsy. They must be read with care; otherwise you may find the road to a solution blocked by an unwarranted assumption.

1.

Mel Stover of Winnipeg was the first to send this amusing problem—amusing because of the ease with which even the best of geometers may fail to approach it properly. Given a triangle with one obtuse angle, is it possible to cut the triangle into smaller triangles, all of them acute? (An acute triangle is a triangle with three acute angles. A right angle is of course neither acute nor obtuse.) If this cannot be done, give a proof of impossibility. If it can be done, what is the smallest number of acute triangles into which any obtuse triangle can be dissected?

The illustration at right shows a typical attempt that leads nowhere. The triangle has been divided into three acute triangles, but the fourth is obtuse, so nothing has been gained by the preceding cuts.

This delightful problem led me to ask myself: "What is the smallest number of acute triangles into which a square can be dissected?" For days I was convinced that nine was the answer; then suddenly I saw how to reduce it to eight. I wonder how many readers can discover an eight-triangle solution, or perhaps an even better one. I am unable to prove that eight is the minimum, though I strongly suspect that it is.

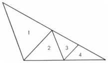
2.

In H. G. Wells's novel *The First Men in the Moon* our natural satellite is found to be inhabited by intelligent insect creatures who live in caverns below the surface. These creatures, let us assume, have a unit of distance that we shall call a "lunar." It was adopted because the moon's surface area, if expressed in square lunars, exactly equals the moon's volume in cubic lunars. The moon's diameter is 2,160 miles. How many miles long is a lunar?

3.

In 1958 John H. Fox, Jr., of the Minneapolis-Honeywell Regulator Co., and L. Gerald Marnie of the Massachusetts Institute of Technology devised an unusual betting game which they call Googol. It is played as follows: Ask someone to take as many slips of paper as he pleases, and on each slip write a different positive number. The numbers may range from small fractions of one to a number the size of a "googol" (1 followed by a hundred zeros) or even larger. These slips are turned face-down and shuffled over the top of a table. One at a time you turn the slips face-up. The aim is to stop turning when you come to the number that you guess to be the largest of the series. You cannot go back and pick a previously turned slip. If you turn over all the slips, then of course you must pick the last one turned.

Most people will suppose the odds



Can this triangle be cut into acute ones?

Secretary Problem

- n *unknown* values $w_1, \dots, w_n$
- Random order
- Step i:
  1. Select w_i and stop
  2. Ignore w_i and continue

$\Pr[$We select $\max_i w_i]$?

Secretary Problem

S_1
Sampling Phase

S_2
Selection Phase

1

S_1
Sampling Phase

S_2
Selection Phase

Secretary Problem

1 5 ...

S_1
Sampling Phase

S_2
Selection Phase

Secretary Problem

S_1
Sampling Phase

S_2
Selection Phase

Secretary Problem

S_1
Sampling Phase

S_2
Selection Phase

Secretary Problem

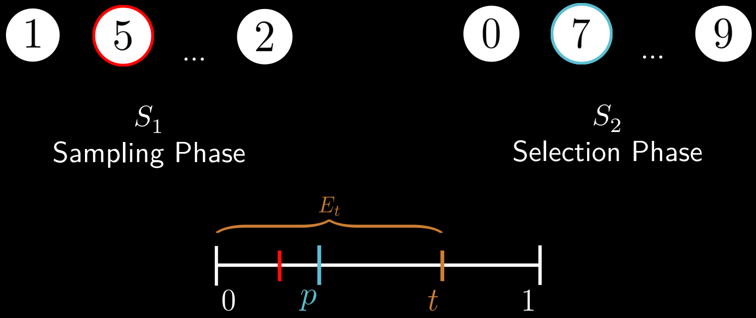
Secretary Problem

1 5 … 2 | 0 7 … 9

S_1 Sampling Phase | S_2 Selection Phase

$$\left.\begin{array}{ll}\text{w.p. } 1/2, & w_1^* \in S_2 \\ \text{w.p. } 1/2, & w_2^* \in S_1\end{array}\right\} \implies Pr[\text{We select } \max_i w_i] \geq 1/4$$

Secretary Problem



- Optimal: fix arrival time t of w_1^*

$$Pr[ALG \leftarrow w_1^*] = \int_p^1 \Pr\left[\text{Largest element in } E_t \in S_1\right] dt$$

$$= \int_p^1 \frac{p}{t}\, dt = p \ln\left(\frac{1}{p}\right) = {}^1\!/_e \text{ for } p = {}^1\!/_e$$

Generalization

Matroid Secretary Problem

For a matroid $M = (E, \mathcal{I})$ and (unknown) weights $w : E \to \mathbb{R}$, select $S \subseteq E$

- online in uniformly random order,
- $S \in \mathcal{I}$ (*independent*),
- to *maximize* $w(S) = \sum_{e \in S} w_e$

Compare against $OPT = \max_{T \in \mathcal{I}} w(T)$

Matroid Secretary Conjecture

Matroid Secretary Conjecture
[Babaioff, Immorlica, Kleinberg '07]

$\exists c > 0$ s.t. $\forall$ matroid M and weights $w : E \to \mathbb{R}$,
$\exists$ algorithm $\mathcal{A}_M$ for MSP s.t.

$$\mathbb{E}\left[w\left(\mathcal{A}_M\right)\right] \geq c \cdot OPT_M$$

Matroid Secretary Conjecture

Matroid Secretary Conjecture
[Babaioff, Immorlica, Kleinberg '07]

$\exists c > 0$ s.t. $\forall$ matroid M and weights $w : E \to \mathbb{R}$,
$\exists$ algorithm $\mathcal{A}_M$ for MSP s.t.

$$\mathbb{E}\left[w\left(\mathcal{A}_M\right)\right] \geq c \cdot OPT_M$$

Strong Matroid Secretary Conjecture

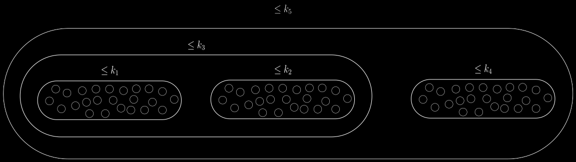
The Matroid Secretary Conjecture holds for $c = 1/e$ for all matroids

State of the Art

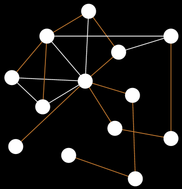
- Transversal Matroids: $\frac{1}{e}$-approx.
  [Kesselheim, Radke, Tönnis, Vö '13]
- Laminar Matroids: $\frac{1}{4.75}$-approx.
  [Huang, Parsaeian, Zhu 24']
- Graphic Matroids: $\frac{1}{4}$-approx.
  [Soto, Turkieltaub, Verdugo '18]
- Co-graphic Matroids: $\frac{1}{3e}$-approx.
  [Soto '13]
- Regular Matroids: $\frac{1}{9e}$-approx.
  [Dinitz, Kortsarz '14]
- General Matroids: $\Omega\left(\frac{1}{\log\log r}\right)$-approx.
  [Lachish '14]
- Binary Matroids, Gammoids: $\Omega(1)$ is open!

What We Study

Laminar Matroid



Graphic Matroid



Algorithm: Greedy Improving

Fix a "sampling" parameter p.

Greedy Improving Algorithm (p)

- $S \leftarrow \emptyset$
- For $i \leftarrow 1$ to $\lceil p\, n \rceil$
  - Skip i
- For $i \leftarrow \lceil p\, n \rceil + 1$ to n
  - Observe w_i
  - If $S + i \in \mathcal{I}$ and $i \in OPT_{\leq i}$
    - $S \leftarrow S + i$
- Return S

Past Work: Laminar Matroids

- $3/16000$-approx.
  [Im, Wang '11]
- 0.07-approx.
  [Jaillet, Soto, Zenklusen '13]
- 0.104-approx.
  [Ma, Tang, Wang '13]
- 0.192-approx.
  [Soto, Turkieltaub, Verdugo '18]
- 0.210-approx.
  [Huang, Parsaeian, Zhu '24]

Past Work: Laminar Matroids

- $3/16000$-approx.
  [Im, Wang '11]
- 0.07-approx.
  [Jaillet, Soto, Zenklusen '13]
- 0.104-approx.
  [Ma, Tang, Wang '13]
- 0.192-approx.
  [Soto, Turkieltaub, Verdugo '18]
- 0.210-approx.
  [Huang, Parsaeian, Zhu '24]

- Greedy Improving Algorithm

Our Contributions

1. Optimal analysis of Greedy Improving algorithm for laminar matroids:
   $1 - \ln(2) \approx 0.306$-approx.

Our Contributions

1. Optimal analysis of Greedy Improving algorithm for laminar matroids:
   $1 - \ln(2) \approx 0.306$-approx.
   (Need different algorithms for strong MSC)

Our Contributions

1. Optimal analysis of Greedy Improving algorithm for laminar matroids:
   $1 - \ln(2) \approx 0.306$-approx.
   (Need different algorithms for strong MSC)
2. Improved algorithm for graphic matroids:
   $0.25 \implies 0.2504$-approx.

Our Contributions

1. Optimal analysis of Greedy Improving algorithm for laminar matroids:
   $1 - \ln(2) \approx 0.306$-approx.
   (Need different algorithms for strong MSC)
2. Improved algorithm for graphic matroids:
   $0.25 \implies 0.2504$-approx.
3. Improved algorithm for rank-2 matroids (all laminar), which beats Greedy Improving (0.3462-approx)

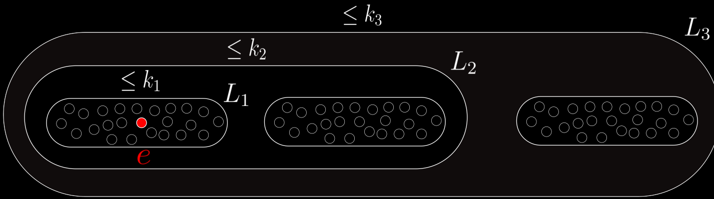
Our Contributions

1. Optimal analysis of Greedy Improving algorithm for laminar matroids:
   $1 - \ln(2) \approx 0.306$-approx.
   (Need different algorithms for strong MSC)
2. Improved algorithm for graphic matroids:
   $0.25 \implies 0.2504$-approx.
3. Improved algorithm for rank-2 matroids (all laminar), which beats Greedy Improving (0.3462-approx)
4. All via a unified *labeling scheme* technique, which subsumes past work

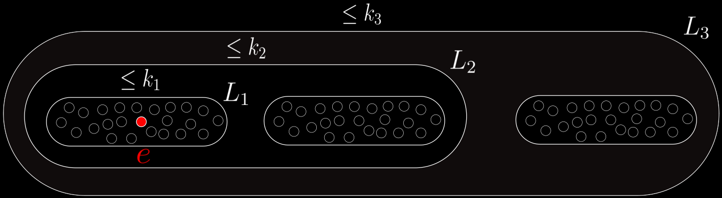
Our Contributions

1. Optimal analysis of Greedy Improving algorithm for laminar matroids:
   $1 - \ln(2) \approx 0.306$-approx.
   (Need different algorithms for strong MSC)
2. Improved algorithm for graphic matroids:
   $0.25 \implies 0.2504$-approx.
3. Improved algorithm for rank-2 matroids (all laminar), which beats Greedy Improving (0.3462-approx)
4. All via a unified *labeling scheme* technique, which subsumes past work

Key Issue



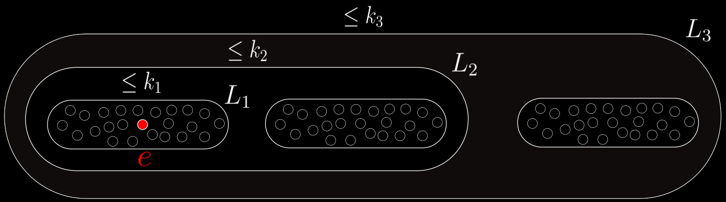
Key Issue



- Want to calculate

$$\Pr[\,\exists \text{ space for } e\,] =$$
$$\Pr[\,|S \cap L_1| \leq k_1 - 1 \ \wedge\ |S \cap L_2| \leq k_2 - 1 \ \wedge\ \ldots]$$

Key Issue



- Want to calculate

$$\Pr[\,\exists \text{ space for } e\,] =$$
$$\Pr[\,|S \cap L_1| \leq k_1 - 1 \;\wedge\; |S \cap L_2| \leq k_2 - 1 \;\wedge\; \ldots]$$

- Computing $\Pr[\,|S \cap L_i| \leq k_i - 1\,]$ is easy but

$$|S \cap L_i| \leq k_i - 1 \quad \text{and} \quad |S \cap L_j| \leq k_j - 1$$

are correlated events

What Does an Optimal Analysis Look Like?

- Assume e arrives at $t_e \sim U[0, 1]$

What Does an Optimal Analysis Look Like?

e arriving at time t is improving $\iff$ $e \in OPT_{\leq t}$

- Assume e arrives at $t_e \sim U[0, 1]$
- $N[a, b)$: # of improving elements in $[a, b)$

What Does an Optimal Analysis Look Like?

e arriving at time t is improving $\iff$ $e \in OPT_{\leq t}$

- Assume e arrives at $t_e \sim U[0,1]$
- $N[a,b)$: # of improving elements in $[a,b)$

We show $N[a,b) \sim Poi\left(r \cdot \ln(b/a)\right)$

What Does an Optimal Analysis Look Like?

e arriving at time t is improving $\iff e \in OPT_{\leq t}$

- Assume e arrives at $t_e \sim U[0,1]$
- $N[a,b)$: # of improving elements in $[a,b)$

We show $N[a,b) \sim Poi\left(r \cdot \ln(b/a)\right)$

- $S(t)$: last *improving* element in $[0,t)$

$$\Pr[S(b) \leq x] = \prod_{e \in OPT(E_b)} \Pr[t_e \leq x] = \left(\frac{x}{b}\right)^r$$

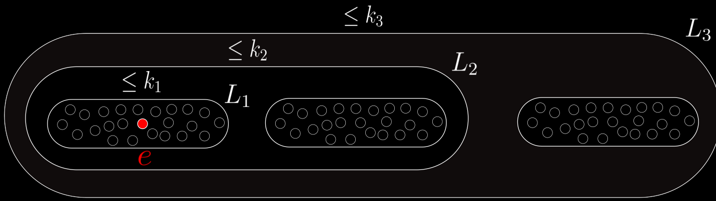
- $y_0 = 1, y_k = S(y_{k-1})$. Also, $x_k \triangleq -\ln(y_k)$.

$$\begin{aligned}\Pr[x_k - x_{k-1} \leq x] &= \Pr\left[\ln(y_{k-1}) - \ln(y_k) \leq x\right] \\ &= \Pr\left[S(y_{k-1}) \geq y_{k-1}e^{-x}\right] = 1 - \left(\frac{y_{k-1}e^{-x}}{y_{k-1}}\right)^r \\ &= 1 - e^{-xr}\end{aligned}$$

Labeling Scheme

- Fix $e \in OPT$. e is selected iff

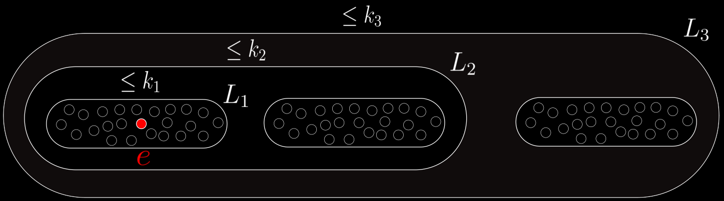
$$|S \cap L_1| \leq k_1 - 1 \;\; \wedge \;\; |S \cap L_2| \leq k_2 - 1 \;\; \wedge \;\; \ldots$$



Labeling Scheme

- Fix $e \in OPT$. e is selected iff

$$|S \cap L_1| \leq k_1 - 1 \;\wedge\; |S \cap L_2| \leq k_2 - 1 \;\wedge\; \dots$$

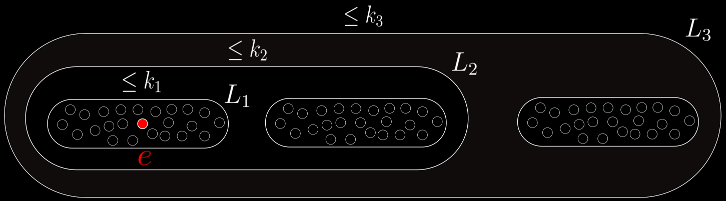


- Set aside a special label for $e \implies \ell(e) = 1$

Labeling Scheme

- Fix $e \in OPT$. e is selected iff

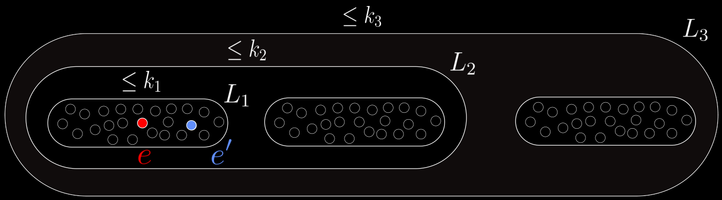
$$|S \cap L_1| \leq k_1 - 1 \;\wedge\; |S \cap L_2| \leq k_2 - 1 \;\wedge\; \ldots$$



- Set aside a special label for $e \implies \ell(e) = 1$

Main Idea

Let $e \in L_1 \subseteq L_2 \subseteq \ldots \subseteq L_m$
At each *improving element* e', assign a label $\ell(e')$:

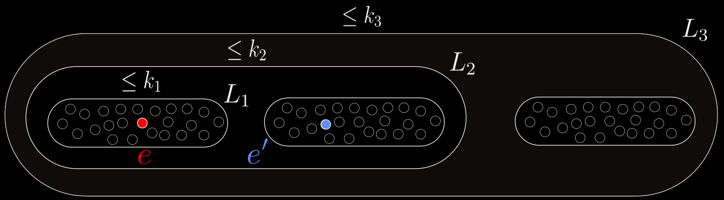


Main Idea

Let $e \in L_1 \subseteq L_2 \subseteq \ldots \subseteq L_m$
At each *improving element* e', assign a label $\ell(e')$:

1. $e' \in L_1 \implies \ell(e') = k_1$

Labeling Scheme

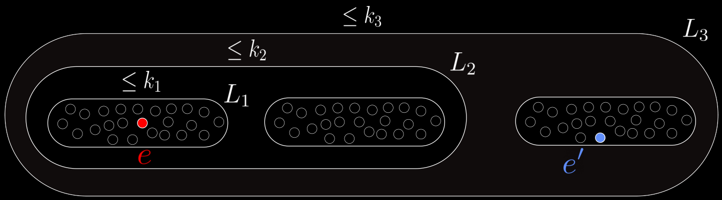


Main Idea

Let $e \in L_1 \subseteq L_2 \subseteq \ldots \subseteq L_m$
At each *improving element* e', assign a label $\ell(e')$:

1. $e' \in L_1 \implies \ell(e') = k_1$
2. $e' \in L_2 \setminus L_1 \implies \ell(e') = k_2$

Labeling Scheme



Main Idea

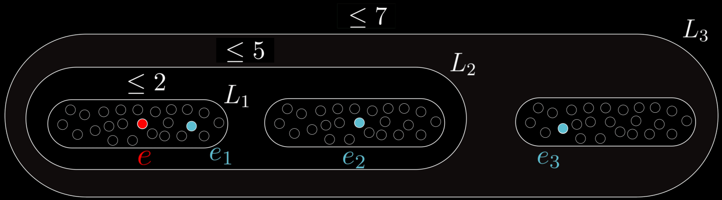
Let $e \in L_1 \subseteq L_2 \subseteq \ldots \subseteq L_m$
At each *improving element* e', assign a label $\ell(e')$:

1. $e' \in L_1 \implies \ell(e') = k_1$
2. $e' \in L_2 \setminus L_1 \implies \ell(e') = k_2$
3. $e' \in L_3 \setminus L_2 \implies \ell(e') = k_3$

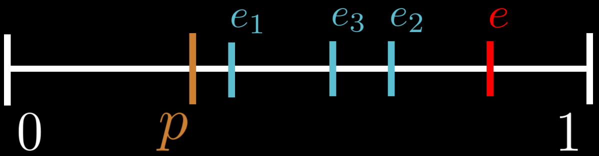
$\vdots$

$\implies$ $z^e = \ell(e_1) \cdot \ell(e_2) \cdot \ldots$ is the improving word of e

Improving Word



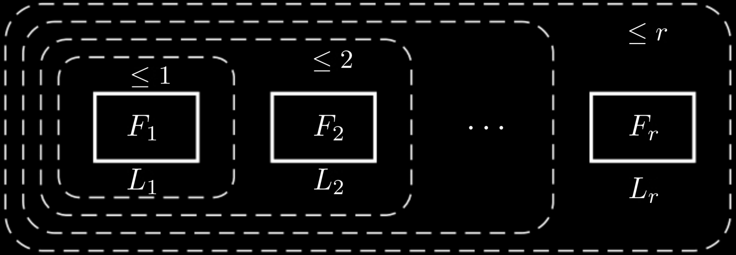
$z^e = \ell(e_1) \cdot \ell(e_2) \cdot \ldots$ is the improving word of e



$$z = 2\ 7\ 5\ 1$$

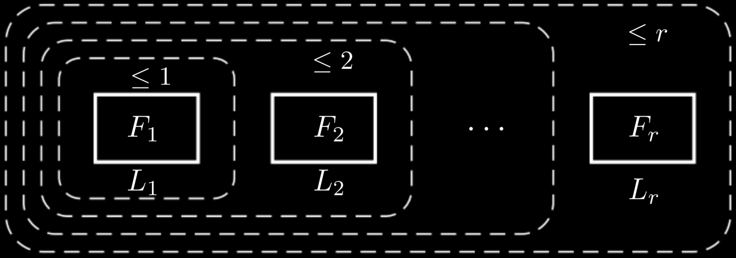
Improving Word

In the worst case,



Improving Word

In the worst case,

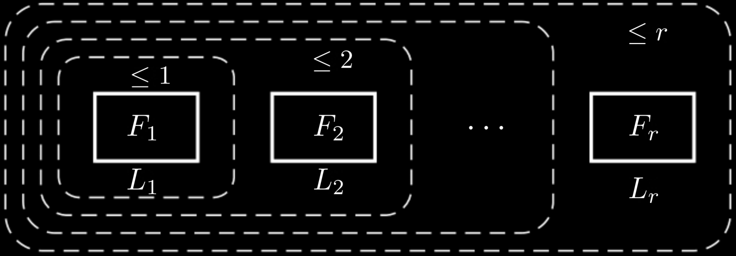


Let $z^e = y^e 1 x$, i.e. y^e denotes the labels of improving elements before e. For e to be selected, we want

- $|\{i \mid y_i^e \leq 1\}| \leq 0$
- $|\{i \mid y_i^e \leq 2\}| \leq 1$
- $|\{i \mid y_i^e \leq 3\}| \leq 2$

Improving Word

In the worst case,



Let $z^e = y^e 1x$, i.e. y^e denotes the labels of improving elements before e. For e to be selected, we want

- $|\{i \mid y_i^e \leq 1\}| \leq 0$
- $|\{i \mid y_i^e \leq 2\}| \leq 1$
- $|\{i \mid y_i^e \leq 3\}| \leq 2$

$\implies$ We want $\forall j \in [r]$,

$$|\{i \mid y_i^e \leq j\}| \leq j - 1$$

Language of a Matroid

$\forall M$, create $\mathcal{L}_M$ s.t. $\forall e \in OPT$

$$\Pr[e \in ALG] \geq \Pr\left[y^e \in \mathcal{L}_M\right]$$

Language of a Matroid

$\forall M$, create $\mathcal{L}_M$ s.t. $\forall e \in OPT$

$$\Pr[e \in ALG] \geq \Pr\left[y^e \in \mathcal{L}_M\right]$$

Examples:

- Uniform:

$$\mathcal{L}_M = \{y1x \in [r]^* \mid x \in ([r]-1)^* \text{ and } |y| \leq r-1\}$$

Language of a Matroid

$\forall M$, create $\mathcal{L}_M$ s.t. $\forall e \in OPT$

$$\Pr[e \in ALG] \geq \Pr\left[y^e \in \mathcal{L}_M\right]$$

Examples:

- Uniform:

$$\mathcal{L}_M = \{y1x \in [r]^* \mid x \in ([r]-1)^* \text{ and } |y| \leq r-1\}$$

- Laminar:

$$\mathcal{L}_M = \{y1x \in [r]^* \mid x \in ([r]-1)^* \text{ and } \forall 1 \leq j \leq r,\ |\{i \mid y_i \leq j\}| \leq j-1\}$$

Language of a Matroid

$\forall M$, create $\mathcal{L}_M$ s.t. $\forall e \in OPT$

$$\Pr[e \in ALG] \geq \Pr\left[y^e \in \mathcal{L}_M\right]$$

Examples:

- Uniform:

$$\mathcal{L}_M = \{y1x \in [r]^* \mid x \in ([r]-1)^* \text{ and } |y| \leq r-1\}$$

- Laminar:

$$\mathcal{L}_M = \{y1x \in [r]^* \mid x \in ([r]-1)^* \text{ and } \forall 1 \leq j \leq r,\ |\{i \mid y_i \leq j\}| \leq j-1\}$$

- Graphic:

Language of a Matroid

$\forall M$, create $\mathcal{L}_M$ s.t. $\forall e \in OPT$

$$\Pr[e \in ALG] \geq \Pr\left[y^e \in \mathcal{L}_M\right]$$

Examples:

- Uniform:

$$\mathcal{L}_M = \{y1x \in [r]^* \mid x \in ([r]-1)^* \text{ and } |y| \leq r-1\}$$

- Laminar:

$$\mathcal{L}_M = \{y1x \in [r]^* \mid x \in ([r]-1)^* \text{ and } \forall 1 \leq j \leq r,\ |\{i \mid y_i \leq j\}| \leq j-1\}$$

- Graphic: ...even more complicated!

Result for Laminar Matroids

1. Labeling scheme is independent of arrival order $\implies$ Conditioned on $|z^e| = k$, z is a uniformly random word from $[r]^k$

Result for Laminar Matroids

1. Labeling scheme is independent of arrival order $\implies$ Conditioned on $|z^e| = k$, z is a uniformly random word from $[r]^k$
2. For $z^e = y^e 1 x$, conditioned on $|y^e| = m \leq r$, $(1)+$ induction on $m \implies$

$$\Pr\left[\bigwedge_{j=1}^{r} |\{i \mid y_i^e \leq j\}| \leq j-1\right] = 1 - \frac{m}{r}$$

Result for Laminar Matroids

1. Labeling scheme is independent of arrival order $\implies$ Conditioned on $|z^e| = k$, z is a uniformly random word from $[r]^k$
2. For $z^e = y^e 1 x$, conditioned on $|y^e| = m \leq r$, $(1)+$ induction on m $\implies$

$$\Pr\left[\bigwedge_{j=1}^{r} |\{i \mid y_i^e \leq j\}| \leq j-1\right] = 1 - \frac{m}{r}$$

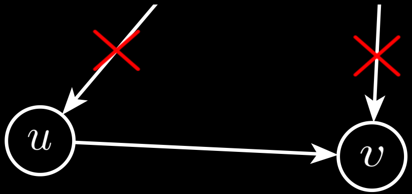
3. $(1) + (2)+$ the number of improving elements (i.e. $|z^e|$) follows a Poisson distribution $\implies$

$$\Pr\left[y^e \in \mathcal{L}_M\right] \geq 1 - \ln(2) \approx 0.3068$$

Labeling Scheme for Graphic Matroids

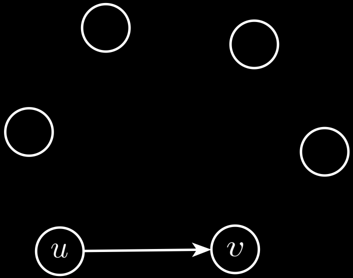
High Level Idea

Past algorithms:
Design orientation of E s.t. $\forall$ improving $e = (u, v)$, take e if $\deg^{-}(u) = \deg^{-}(v) = 0$



High Level Idea

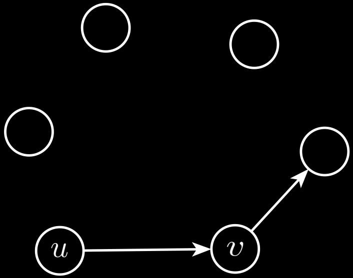
Wrong Approach:
Design orientation of E s.t. $\forall$ improving $e = (u, v)$, take e if $\deg^{-}(v) = 0$



Labeling Scheme for Graphic Matroids

High Level Idea

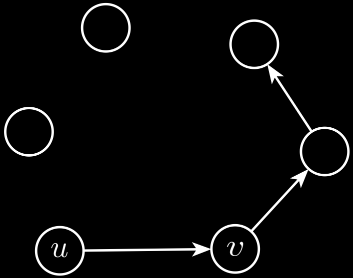
Wrong Approach:
Design orientation of E s.t. $\forall$ improving $e = (u, v)$, take e if $\deg^{-}(v) = 0$



Labeling Scheme for Graphic Matroids

High Level Idea

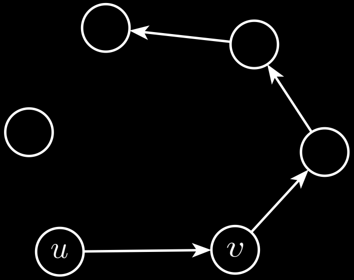
Wrong Approach:
Design orientation of E s.t. $\forall$ improving $e = (u, v)$, take e if $\deg^{-}(v) = 0$



Labeling Scheme for Graphic Matroids

High Level Idea

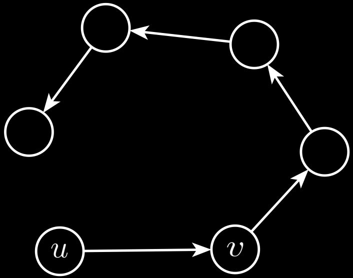
Wrong Approach:
Design orientation of E s.t. $\forall$ improving $e = (u, v)$, take e if $\deg^{-}(v) = 0$



Labeling Scheme for Graphic Matroids

High Level Idea

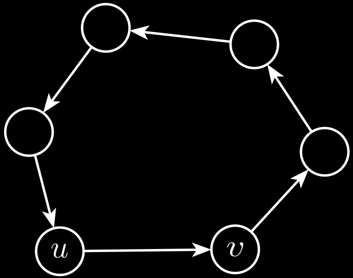
Wrong Approach:
Design orientation of E s.t. $\forall$ improving $e = (u, v)$, take e if $\deg^{-}(v) = 0$



Labeling Scheme for Graphic Matroids

High Level Idea

Wrong Approach:
Design orientation of E s.t. $\forall$ improving $e = (u, v)$, take e if $\deg^{-}(v) = 0$

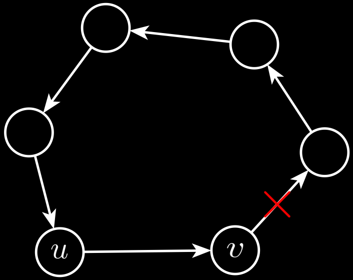


Labeling Scheme for Graphic Matroids

High Level Idea

Correct Approach:

Design orientation of E s.t. $\forall$ improving $e = (u, v)$, take e if $\deg^{-}(v) = 0$ and e is not second in a path of seen edges



$\Longrightarrow$ 0.2504-approx.

$\Longrightarrow$ 0.2693-approx. for simple graphs

Conclusion

- Technique also subsumes prior work on special classes of matroids
- Hopefully can be used on
  - matroid classes for which the conjecture is still open (e.g. gammoids), to give constant-factor algorithms
  - matroid classes for which a constant is known to give a $1/e$-approximation

Thanks!

Questions?