Improved Mechanisms and Prophet Inequalities for Graphical Dependencies

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Joint work with

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Prophet Inequality

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[Krengel, Sucheston and Garling '77]

$X_1, X_2, \ldots, X_n \sim$ (known) $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$
arrive in *adversarial* order.

- Design *stopping time* to maximize selected value.
- Compare against all-knowing *prophet*: $\mathbb{E}[\max_i X_i]$.
- Competitive Ratio:

$$\frac{\mathbb{E}[ALG]}{\mathbb{E}[\max_i X_i]}.$$

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Prophet Inequality [Krengel, Sucheston and Garling '77, '78]

$\exists$ stopping strategy that achieves $1/2 \cdot \mathbb{E}[\max_i X_i]$, and this is tight.

Optimal strategy is a single-threshold algorithm T

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Applications:

- Auction Design
- Revenue Maximization
- Matching Markets
- Resource Allocation
- Portfolio Selection
- Supply Chain Management

Single Buyer Revenue Maximization

- One buyer, n items
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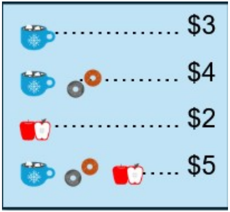
$$k^* = \arg\max_k \mathop{\mathbb{E}}_{S \sim \Gamma_k} [v(S) - p_k]$$

Single Buyer Revenue Maximization



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- Valuations:
  - Additive: $v(S) = \sum_{i \in S} v(i)$
  - Unit-Demand: $v(S) = \max_{i \in S} v(i)$
  - Subadditive: $v(A \cup B) \leq v(A) + v(B), \quad \forall A, B \subseteq [n]$

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Cannot compute optimal menu $\implies$ approximation
[Daskalakis, Deckelbaum, Tzamos '17]

Two Mechanisms:

1. SRev: Maximum revenue obtainable by pricing each item separately.
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Theorem

$\max\{\textsc{SRev}, \textsc{BRev}\}$ is a $\Omega(1)$-approximation to $\textsc{Rev}$ for a subadditive buyer.

[Chawla, Hartline, Kleinberg '07], [Chawla, Malec, Sivan '10], [Babaioff, Immorlica, Lucier, Weinberg '14], [Rubinstein, Weinberg '15]

Correlations

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Previous models:

- Linear Correlations
  [Chawla, Malec, Sivan '10], [Bateni, Dehghani, Hajiaghayi, Seddighin '15], [Immorlica, Singla, Waggoner '20]
- Pairwise-Independent
  [Caragiannis, Gravin, Lu, Wang '22], [Dughmi, Kalayci, Patel '24]

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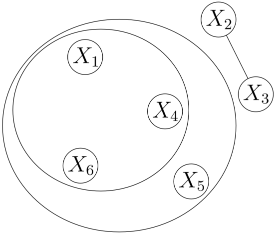
Graphical model of correlation (physics, statistics, ML, ...)



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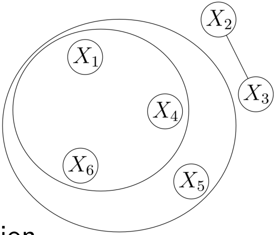

Max. Weighted Degree Δ

- $\Delta = 0 \implies$ independence
- $\Delta = +\infty \implies$ arbitrary (positive) correlation

$$e^{-4\Delta} \le \frac{\Pr\left[X_i = y_i \mid \mathbf{X}_{-\mathbf{i}} = \mathbf{y}_{-\mathbf{i}}\right]}{\Pr[X_i = y_i]} \le e^{4\Delta}, \qquad \forall i, \mathbf{y}$$

[Brustle, Cai, Daskalakis '20]

Theorem [Cai, Oikonomou '21]

$\max\{\textsc{SRev}, \textsc{BRev}\}$ is a $e^{-O(\Delta)}$-approximation to $\textsc{Rev}$ for a additive/unit-demand buyer.

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$\max\{\text{SREV}, \text{BREV}\}$ is a $e^{-O(\Delta)}$ -approximation to REV for a additive/unit-demand buyer.

$\exists$ a $e^{-O(\Delta)}$-competitive prophet inequality for MRFs.

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Upper bound of $O\left(\Delta^{-1/7}\right)$.

Our Results

Theorem [L., Patton, Singla '24]

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Theorem [L., Patton, Singla '24]

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$\exists$ a $\mathrm{O}(1/\Delta)$-competitive prophet inequality for MRFs.

- Exponential improvement.
- Generalization to a subadditive buyer.
- Tight for prophet inequality.

Techniques for Independent Items

1. Core-Tail Decomposition
   - Core: $\mathcal{C} = \{i \mid v(i) \leq T\}$
   - Tail: $\mathcal{T} = \{i \mid v(i) > T\}$

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2. Marginal Mechanism Lemma

$$Rev(D) \leq Val\left(D^{\mathcal{C}}\right) + Rev\left(D^{\mathcal{T}}\right)$$

[Hart, Nisan '12], [Cai, Huang '13], [Rubinstein, Weinberg '15]

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3. Choose "balanced" T
   - $T \nearrow$, T is sparse, $Rev\left(D^{\mathcal{T}}\right) \approx \textsc{SRev}$
   - $T \searrow$, $Val\left(D^{\mathcal{C}}\right)$ concentrates, $Val\left(D^{\mathcal{C}}\right) \approx \textsc{BRev}$

Approximate Marginal Mechanism Lemma

Lemma

Let A, B be disjoint sets of items, and D be an arbitrary distribution over monotone subadditive valuations $v : 2^{A \cup B} \to \mathbb{R}_{\geq 0}$. Then,

$$Rev(D) \leq 2\left(Val\left(D^A\right) + Rev\left(D^B\right)\right).$$

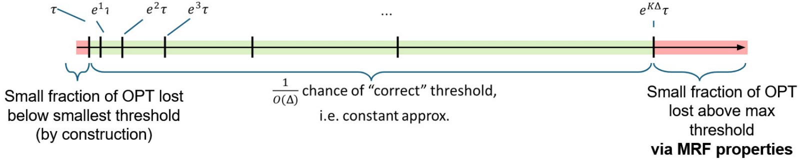
Proof based on discounted prices – restrict $Rev(D)$ to B, discount p_i by $1/2$

Exponential Bucketing

Single Threshold $\tau \implies$ Random Threshold $T \leftarrow \{\tau, e\tau, \dots, e^{k\Delta}\tau\}$

- Prophet Inequalities: $\tau = \mathbb{E}[\max_i X_i]/2$
- Grand Bundle Pricing: $\tau = Val\left(D^{\mathcal{C}}\right)/2$

"Guess" scale of the problem



Questions?