

Oracle-Augmented Prophet Inequalities & How Much You Can Win by Cheating!

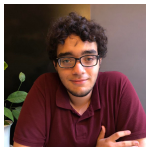
Vasilis Livanos

University of Chile

Joint work with



Sarel
Har-Peled



(El)Farouk
Harb

University of Illinois Urbana-Champaign

ICALP, Tallinn, Estonia, July 12th, 2024









$$X_1 \sim U[4, 6]$$

$$X_4 = \begin{cases} 450 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$$

$$X_2 \sim U[2, 8]$$

$$X_3 \sim U[0, 9]$$



$X_1 \sim U[4, 6]$



4.88



4.88

$$X_4 = \begin{cases} 450 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$$

$$X_2 \sim U[2, 8]$$

$$X_3 \sim U[0, 9]$$





6.67



~~4.88~~

6.67

$X_3 \sim U[0, 9]$

$$X_4 = \begin{cases} 450 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$$



$X_3 \sim U[0, 9]$



3.23



~~4.88~~

~~6.67~~

3.23

$$X_4 = \begin{cases} 450 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$$



$$X_4 = \begin{cases} 450 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$$





~~4.88~~

0

~~6.67~~

~~3.23~~






6.67



$$\mathbb{E} \left[\max_i X_i \right]$$

A conceptual image depicting a saint with a long white beard and a glowing blue halo, dressed in a white robe with a brown sash. He stands in a grand, ornate room with a high, vaulted ceiling and large chandeliers. He is pointing with a thin rod at a large blackboard. The blackboard contains mathematical text. In the foreground, a man in a blue checkered jacket sits at a desk, looking at the blackboard. Other people are visible in the background, seated at similar desks. The scene is lit with warm, golden light from the chandeliers, contrasting with the bright blue glow of the saint's halo and the blackboard's text.

Set $T = 1/2 \cdot \mathbb{E}[\max_i X_i]$
Accept first $X_i \geq T$

Set $T = 1/2 \cdot \mathbb{E}[\max_i X_i]$

Accept first $X_i \geq T$

Set $T = 1/2 \cdot \mathbb{E}[\max_i X_i]$

Accept first $X_i \geq T$

[*Kleinberg, Weinberg*, '12]

Set $T = 1/2 \cdot \mathbb{E}[\max_i X_i]$

Accept first $X_i \geq T$

[*Wittmann*, '95]

[*Kleinberg, Weinberg*, '12]



$$\Pr \left[\text{Gambler} \leftarrow \max_i X_i \right] \geq 1/e$$

(concentration)

$$\Pr \left[\text{Gambler} \leftarrow \max_i X_i \right] \geq 1/e$$

(concentration)

[*Esfandiari, HajiAghayi,*
Lucier, Mitzenmacher,' 20]



Top — 1 — of — k

Top — 1 — of — k

$$\geq 1 - \frac{1}{k+1} \quad [AS'00, AGS'02]$$

Top $1 - \text{of} - k$

$$\geq 1 - \frac{1}{k+1} \quad [AS'00, AGS'02]$$

$$\geq 1 - e^{-k/6}$$

$$\leq 1 - k^{-2k} \quad [EFN'18]$$

Top $1 - \text{of} - k$

$$\geq 1 - \frac{1}{k+1} \quad [AS'00, AGS'02]$$

$$\geq 1 - e^{-k/6}$$

$$\leq 1 - k^{-2k}$$

$[EFN'18]$

Auctions w/ Overbooking









4.88





Questions?





$$\max \frac{\mathbb{E}[\text{ALG}]}{\mathbb{E}[\max_i X_i]}$$



$$\max \Pr \left[\text{ALG} \leftarrow \max_i X_i \right]$$



$$\max \frac{\mathbb{E}[\text{ALG}]}{\mathbb{E}[\max_i X_i]}$$

Oracle_k
 \neq

Top-1-of-(k + 1)



$$\max \Pr \left[\text{ALG} \leftarrow \max_i X_i \right]$$

Oracle_k
 \equiv

Top-1-of-(k + 1)

Oracle₁ \neq Top-1-of-2

$$X_1 = 1 \quad X_2 = \begin{cases} 1 + \varepsilon & \text{w.p. } \frac{1}{2} - \varepsilon \\ 0 & \text{otherwise} \end{cases}$$
$$X_3 = \begin{cases} \frac{1}{\varepsilon} & \text{w.p. } \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Top-1-of-2} \approx 1 + 1 = 2$$

$$\text{Oracle}_1 \approx \frac{1}{2} + 1 = \frac{3}{2}$$



$$\max \frac{\mathbb{E}[\text{ALG}]}{\mathbb{E}[\max_i X_i]}$$

Oracle_k



Top-1-of-(k + 1)



$$\max \Pr \left[\text{ALG} \leftarrow \max_i X_i \right]$$

Oracle_k



Top-1-of-(k + 1)



$$\max \Pr \left[\text{ALG} \leftarrow \max_i X_i \right]$$

$$\text{Oracle}_k \equiv \text{Top-1-of-}(k+1)$$

$$\max \frac{\mathbb{E}[\text{ALG}]}{\mathbb{E}[\max_i X_i]}$$

$$\begin{array}{c} \text{Oracle}_k \\ \neq \end{array}$$

$$\text{Top-1-of-}(k+1)$$

$$\begin{array}{c} \text{ALG} \rightarrow \geq \\ \Downarrow \end{array}$$

$$\text{ALG} \rightarrow \geq$$

Main Results

A square chalkboard with a light-colored wooden frame is centered against a dark, textured background. The words "Main Results" are written in a white, serif font at the top of the board. A thin white horizontal line is drawn directly beneath the text. The rest of the chalkboard surface is dark and shows some faint, light-colored scratches and dust marks.

Main Results

Oracle_k

$$= 1 - e^{-\xi_k} = 1 - e^{-k/e + o(k)}$$

Main Results

Oracle_k

$$= 1 - e^{-\xi_k} = 1 - e^{-k/e + o(k)}$$

Oracle_k(Secretary, IID)

$$\geq 1 - O(k^{-k/5}) \quad \leq 1 - O(k^{-k})$$

Main Results

Oracle_k

$$= 1 - e^{-\xi_k} = 1 - e^{-k/e + o(k)}$$

Oracle_k(Secretary, IID)

$$\geq 1 - O(k^{-k/5}) \quad \leq 1 - O(k^{-k})$$

Top-1-of- k (Secretary, IID)

$$\geq 1 - O(k^{-k/5}) \quad \leq 1 - O(k^{-k})$$

Techniques

Sharding

$$X_i \rightarrow \max\{Y_1, \dots, Y_\ell\} (IID)$$

Techniques

Sharding

$$X_i \rightarrow \max\{Y_1, \dots, Y_\ell\} (IID)$$

Poissonization

$$\max\{Y_1, \dots, Y_\ell\} \equiv \text{Poi}(\lambda)$$

Techniques

Sharding

$$X_i \rightarrow \max\{Y_1, \dots, Y_\ell\} (IID)$$

Poissonization

$$\max\{Y_1, \dots, Y_\ell\} \equiv \text{Poi}(\lambda)$$

Stochastic Dominance

$$\forall x \quad \Pr[\text{ALG} \geq x] \geq c \cdot \Pr[\text{Prophet} \geq x]$$

Upper Bound

$$X_1 = 1 \quad X_2 = \begin{cases} 1 + \varepsilon & \text{w.p. } \frac{1}{2} - \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$X_3 = \begin{cases} \frac{1}{\varepsilon} & \text{w.p. } \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow 1 - \frac{1}{2^{k+1}}$$

Upper Bound

$$X_1 = 1 \quad X_2, \dots, X_{n-1} \sim \text{Poi}(\xi_k)$$

$$X_n = \begin{cases} \frac{1}{\varepsilon} & \text{w.p. } \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\xi_k \implies \Pr[\text{Poi}(\xi_k) = 0] = \Pr[\text{Poi}(\xi_k) > k]$$

$$\xi_k = \frac{k}{e} + o(k)$$

$$1 - \frac{1}{2^{k+1}} \implies 1 - e^{-\xi_k}$$



$$X_1 \sim U[0, 9]$$

$$X_4 \sim U[0, 9]$$

$$X_2 \sim U[0, 9]$$

$$X_3 \sim U[0, 9]$$

IID Idea

Select all $\geq T$

+ Chernoff bound

max of $\pi_1 \pi_2 \dots \pi_n$

changes $O(\log n)$ times

\implies can set T higher

$$\geq 1 - O(k^{-k/5}) \quad \leq 1 - O(k^{-k})$$

Open Questions

Prophet, Gen

Prophet, IID

Secretary, Gen

Secretary, IID

Oracle \rightarrow $\begin{matrix} \Pi? & \Pi? \\ \Pi? & \Pi? \end{matrix}$

Robustness? \implies ML Predictions



Thank You
Estonia!