Oracle-Augmented Prophet Inequalities
& How Much You Can Win by Cheating!

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Joint work with


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$X_1 \sim U[4,6]$
$X_4 = \begin{cases} 450 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$
$X_2 \sim U[2,8]$
$X_3 \sim U[0,9]$


$X_1 \sim U[4,6]$


4.88

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6.67

4.88
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$X_3 \sim U[0, 9]$


3.23

4.88
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0


4.88
0
6.67
3.23


6.67


$\mathbb{E}\left[\max_i X_i\right]$


Set $T = 1/2 \cdot \mathbb{E}[\max_i X_i]$
Accept first $X_i \geq T$

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$[Kleinberg, Weinberg, \ '12]$

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Accept first $X_i \geq T$

$[Wittmann,\ '95]$

$[Kleinberg, Weinberg,\ '12]$


max
Pr [Gambler ← max_i X_i]
?

$$\Pr\left[\text{Gambler} \leftarrow \max_i X_i\right] \geq 1/e$$

$$(concentration)$$

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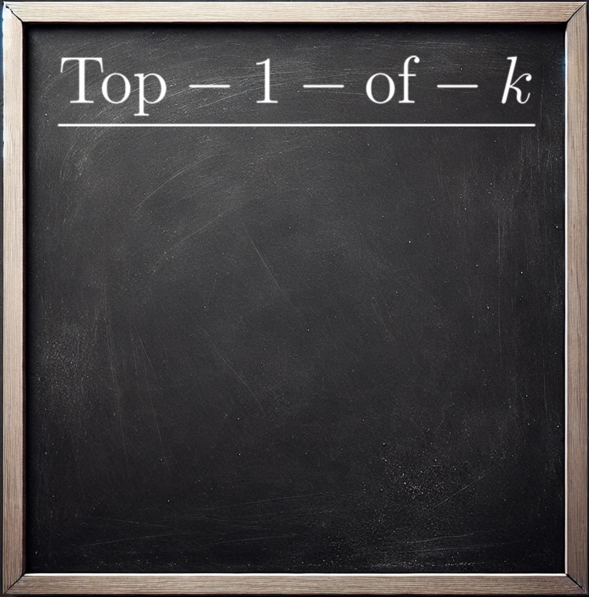
$(concentration)$

[Esfandiari, HajiAghayi, Lucier, Mitzenmacher, '20]


Top − 1 − of − k

$$\mathrm{Top} - 1 - \mathrm{of} - k$$

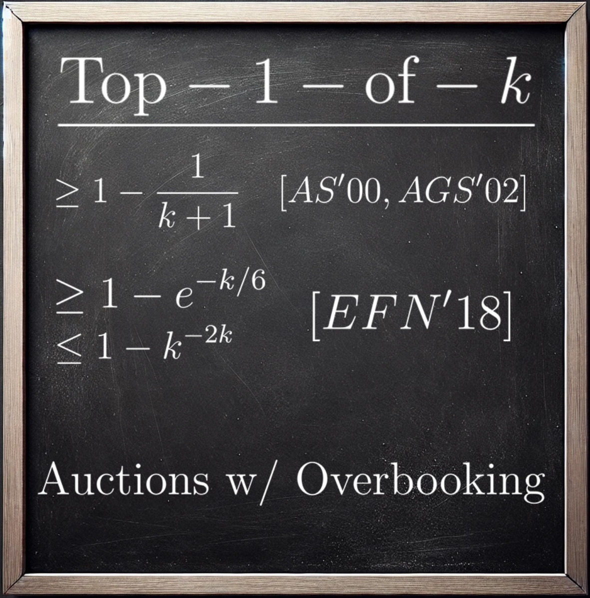
$$\geq 1 - \frac{1}{k+1} \quad [AS'00, AGS'02]$$

$\text{Top} - 1 - \text{of} - k$

$$\geq 1 - \frac{1}{k+1} \quad [AS'00, AGS'02]$$

$$\geq 1 - e^{-k/6} \quad [EFN'18]$$

$$\leq 1 - k^{-2k}$$


Top − 1 − of − k
$\geq 1 - \frac{1}{k+1}$ [AS'00, AGS'02]
$\geq 1 - e^{-k/6}$
$\leq 1 - k^{-2k}$
[EFN'18]
Auctions w/ Overbooking


$X_1 \sim U[4, 6]$

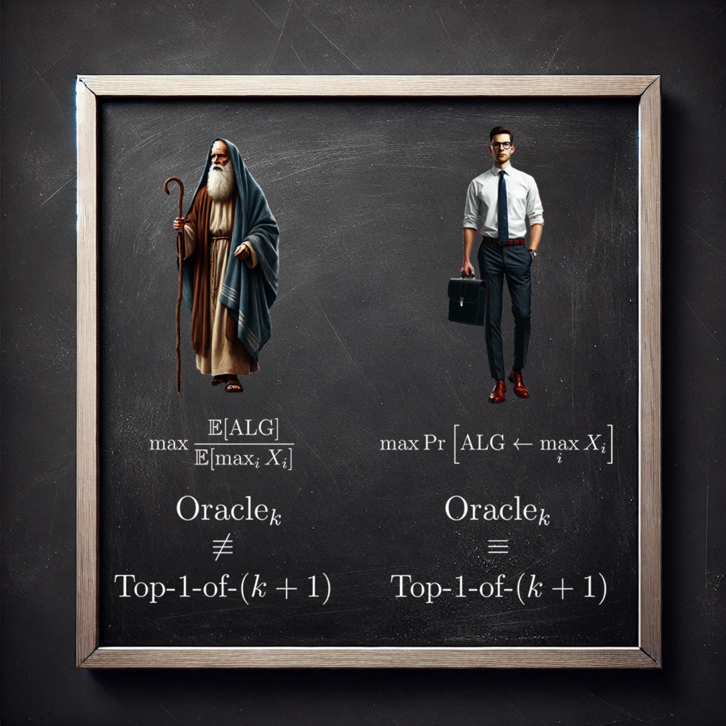

4.88


6.67


Questions?

$$\max \frac{\mathbb{E}[\mathrm{ALG}]}{\mathbb{E}[\max_i X_i]}$$

$$\max \Pr\left[\mathrm{ALG} \leftarrow \max_i X_i\right]$$



$\max \frac{\mathbb{E}[\mathrm{ALG}]}{\mathbb{E}[\max_i X_i]}$
$\max \Pr\left[\mathrm{ALG} \leftarrow \max_i X_i\right]$
Oracle_k
$\not\equiv$
Top-1-of-$(k+1)$
Oracle_k
$\equiv$
Top-1-of-$(k+1)$

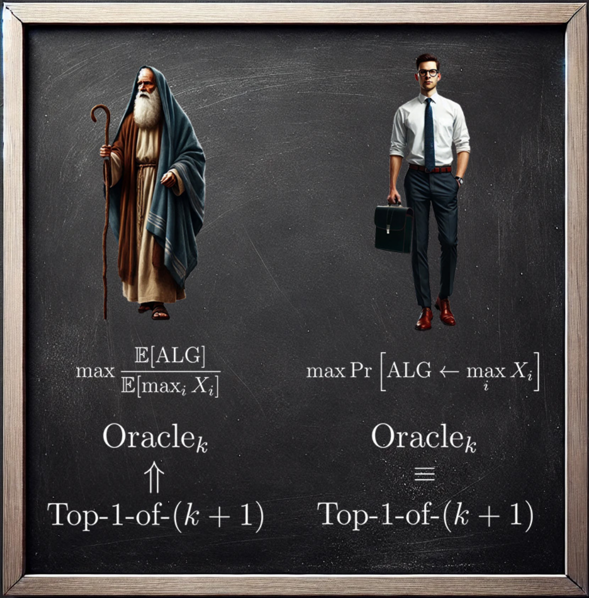
$\text{Oracle}_1 \neq \text{Top-1-of-2}$

$$X_1 = 1 \quad X_2 = \begin{cases} 1+\varepsilon & \text{w.p. } \frac{1}{2} - \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$X_3 = \begin{cases} \frac{1}{\varepsilon} & \text{w.p. } \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Top-1-of-2} \approx 1 + 1 = 2$$

$$\text{Oracle}_1 \approx \frac{1}{2} + 1 = \frac{3}{2}$$



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Oracle_k
$\Uparrow$
Top-1-of-$(k+1)$
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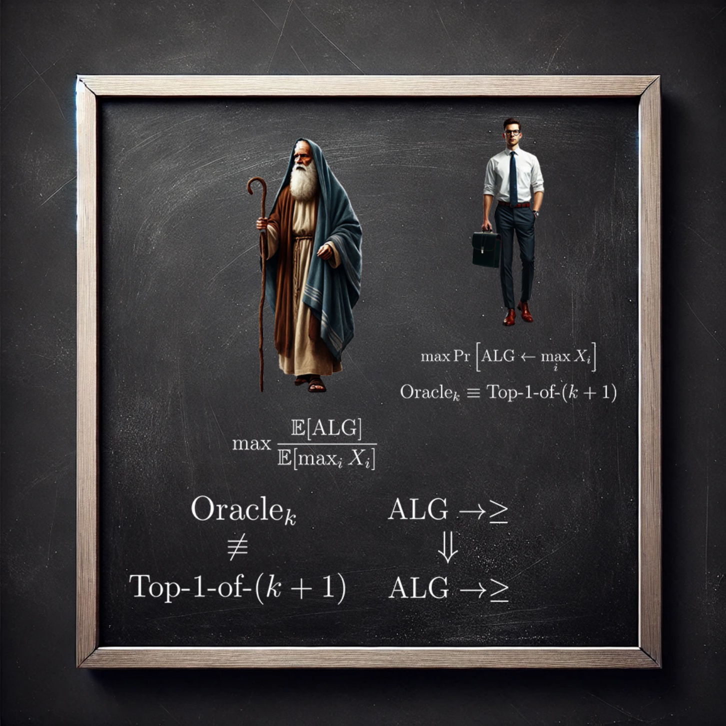
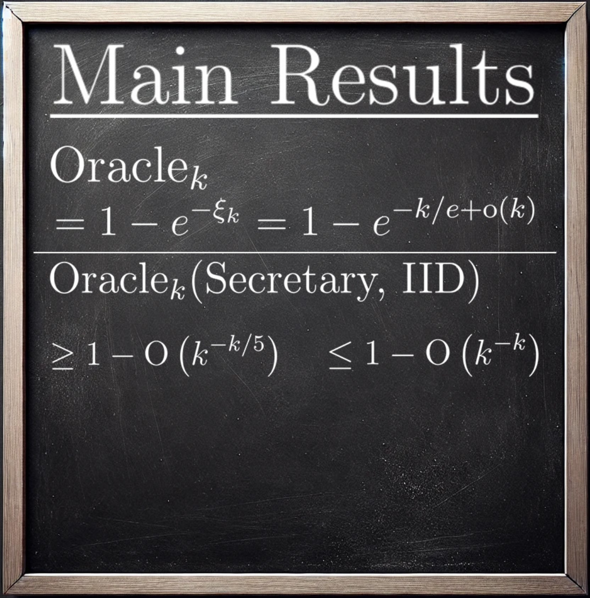
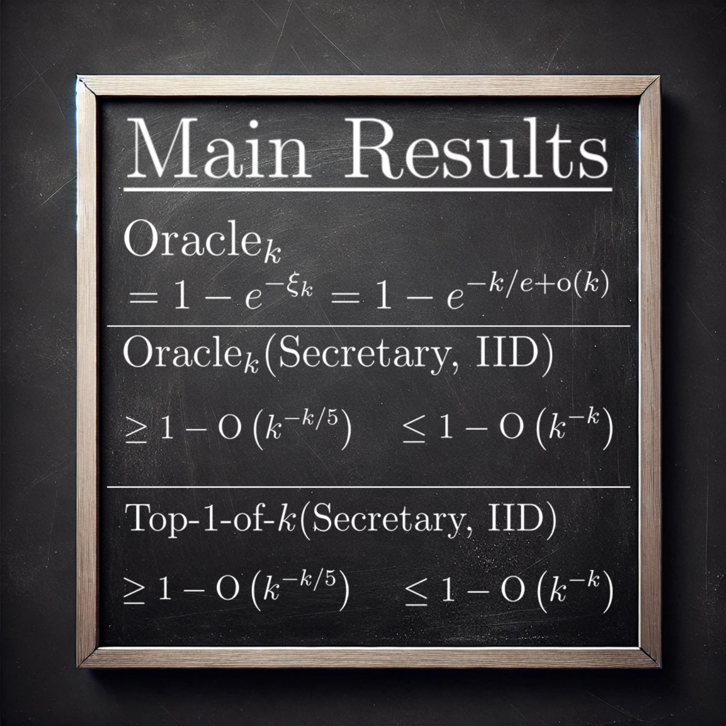

$\max \Pr\left[\text{ALG} \leftarrow \max_i X_i\right]$
$\text{Oracle}_k \equiv \text{Top-1-of-}(k+1)$
$\max \frac{\mathbb{E}[\text{ALG}]}{\mathbb{E}[\max_i X_i]}$
Oracle_k
$\not\equiv$
$\text{Top-1-of-}(k+1)$
$\text{ALG} \rightarrow \geq$
$\Downarrow$
$\text{ALG} \rightarrow \geq$

Main Results

Main Results

$$\mathrm{Oracle}_k = 1 - e^{-\xi_k} = 1 - e^{-k/e + o(k)}$$


Main Results
Oracle_k
$= 1 - e^{-\xi_k} = 1 - e^{-k/e + o(k)}$
$\text{Oracle}_k(\text{Secretary, IID})$
$\geq 1 - O\left(k^{-k/5}\right) \quad \leq 1 - O\left(k^{-k}\right)$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$\geq 1 - O\left(k^{-k/5}\right)$ $\leq 1 - O\left(k^{-k}\right)$
Top-1-of-k(Secretary, IID)
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Techniques

Sharding

$$X_i \to \max\{Y_1, \ldots, Y_\ell\} (IID)$$

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Poissonization

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Sharding

$$X_i \to \max\{Y_1, \ldots, Y_\ell\}(IID)$$

Poissonization

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Stochastic Dominance

$$\forall x \quad \Pr[\mathrm{ALG} \geq x] \geq c \cdot \Pr[\mathrm{Prophet} \geq x]$$

Upper Bound

$$X_1 = 1 \quad X_2 = \begin{cases} 1+\varepsilon & \text{w.p. } \frac{1}{2}-\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$X_3 = \begin{cases} \frac{1}{\varepsilon} & \text{w.p. } \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\implies 1 - \frac{1}{2^{k+1}}$$

Upper Bound

$$X_1 = 1 \quad X_2, \ldots, X_{n-1} \sim \mathrm{Poi}(\xi_k)$$

$$X_n = \begin{cases} \frac{1}{\varepsilon} & \text{w.p. } \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\xi_k \implies \Pr[\mathrm{Poi}(\xi_k) = 0] = \Pr[\mathrm{Poi}(\xi_k) > k]$$

$$\xi_k = \frac{k}{e} + o(k)$$

$$1 - \frac{1}{2^{k+1}} \implies 1 - e^{-\xi_k}$$


$X_1 \sim U[0, 9]$
$X_4 \sim U[0, 9]$
$X_2 \sim U[0, 9]$
$X_3 \sim U[0, 9]$

IID Idea

Select all $\geq T$

+Chernoff bound

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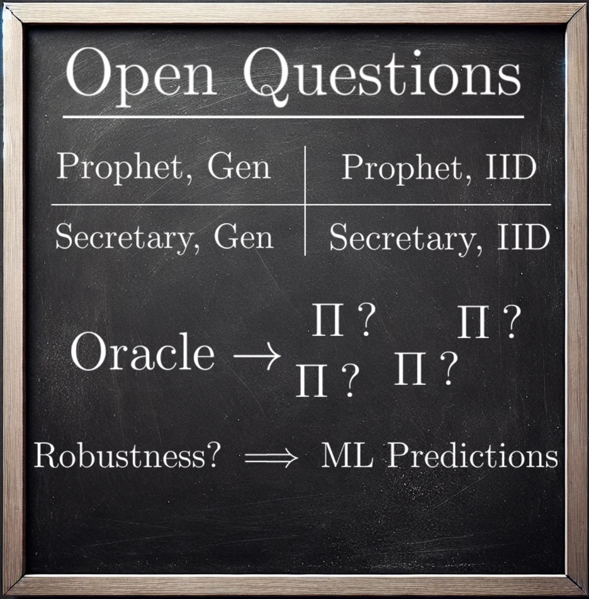
max of $\pi_1 \pi_2 \ldots \pi_n$

changes $O(\log n)$ times

$\implies$ can set T higher

$\geq 1 - O\left(k^{-k/5}\right) \quad \leq 1 - O\left(k^{-k}\right)$


Open Questions
Prophet, Gen
Prophet, IID
Secretary, Gen
Secretary, IID
Oracle → Π ? Π ? Π ? Π ?
Robustness? ⟹ ML Predictions


Thank You
Estonia!