

Threshold Dynamics and Correlated Prophet Inequalities

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Joint work with



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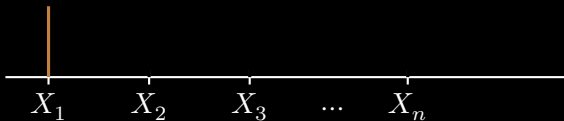
Kevin Schewior



Victor Verdugo

The Prophet Inequality

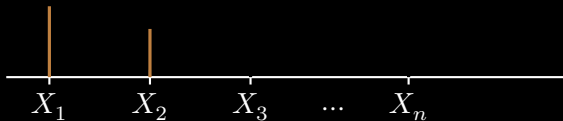
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- ▶ Objective: Maximize selected X_i

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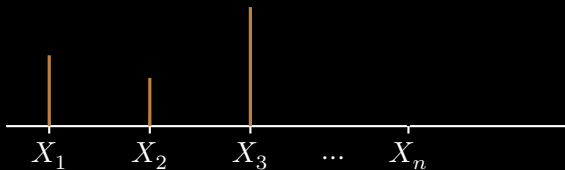
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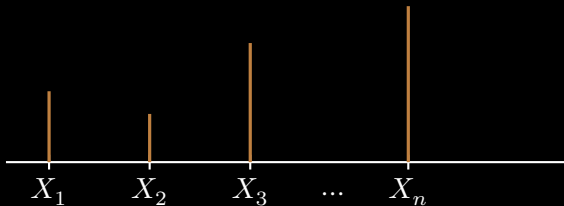
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$$\inf_{F_1, \dots, F_n} \max_{\text{ALG}} \frac{\mathbb{E}[\text{ALG}]}{\mathbb{E}[\max_i X_i]}$$

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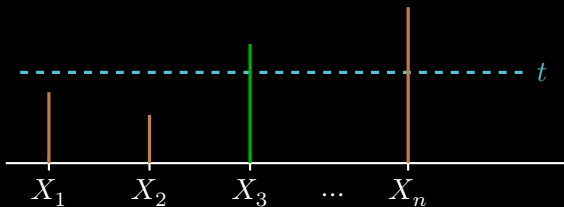
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- ▶ With a single threshold t

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Correlations?

- ▶ If $X_1, \dots, X_n$ negatively correlated $\rightarrow$ **easy**, $\text{CR} \geq 1/2$
[Rinott, Samuel-Cahn '87]
- ▶ (Worst-case) positive correlation $\rightarrow$ **terrible**, $\text{CR} \leq 1/n$
[Hill-Kertz '81]
- ▶ “Structured” correlations studied
 1. Linear correlations [Immorlica, Singla, Waggoner '20]
 2. Graphic models (Markov Random Fields)
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- ▶ This talk: $Z \sim F_Z$ independent of $X_1, \dots, X_n$

Common Base

[Milgrom, Weber '82]

Observe $Y_i = X_i + Z$

Common Scale

Observe $Y_i = X_i \cdot Z$

- ▶ Can get $CR \geq 1/8$ from [ISW '20]

Common Base Model

- Idea: Play threshold t but always select last Y_n
 $V(t) := \text{Value of ALG}$

Shifting Lemma

$$V_Y(t) = \mathbb{E}[Z] + \mathbb{E}[V_X(t - Z)]$$

Proof Sketch:

Condition on $Z = z$ and observe $Y_i \geq t \iff X_i \geq t - z$

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- ▶ Using standard “balanced prices lower bound” $\rightarrow \text{CR} \geq 0.4$

Balanced Prices Lower Bound

$$t \Pr[\max_i X_i \geq t] + (\mathbb{E}[\max_i X_i] - t) \Pr[\max_i X_i < t]$$

$t = 0.8 \mathbb{E}[\max_i X_i]$, and $\text{CR} \geq 0.4$ is tight, if we use BPLB

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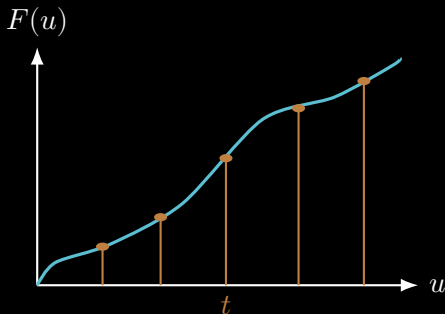
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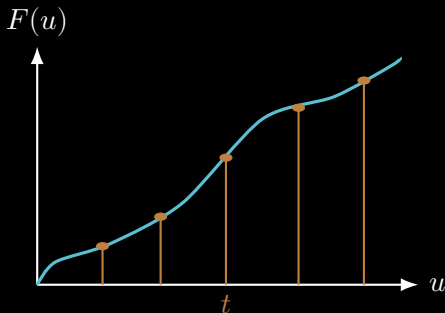
Perhaps, better calibrated t on instance $\rightarrow$ better CR

Classic PI: Searching for the Optimal Threshold

- ▶ Wlog, assume X_i 's supported on $[0, 1]$ – truncate and rescale
- ▶ Parametrize instance by distribution $F := \prod_i F_i$ of OPT
For fixed F , we search for the optimal t
- ▶ Worst-case instance are infinite increasing Bernoullis.



Classic PI: Searching for the Optimal Threshold



► For $t \in [0, 1]$

$$\tilde{X}_t = \begin{cases} t & \text{w.p. } \frac{f(t) dt}{F(t)} \\ 0 & \text{w.p. } 1 - \frac{f(t) dt}{F(t)} \end{cases}$$

Intuitively, $\max_t \tilde{X}_t = \max_i X_i$
but ALG always selects minimum r.v. above t

Main Idea:

$$V(t) = t \Pr[X_t \neq 0] + \Pr[X_t = 0]V(t + dt)$$

Main Idea:

$$V(t) = t \frac{f(t) dt}{F(t)} + \left(1 - \frac{f(t) dt}{F(t)}\right) V(t + dt)$$

Main Idea:

$$-V(t + dt) + V(t) = (t - V(t + dt)) \frac{f(t) dt}{F(t)}$$

Main Idea:

$$-\frac{V(t+dt) - V(t)}{dt} = (t - V(t+dt)) \frac{f(t)}{F(t)}$$

ODE

$$V'(t) = (V(t) - t) \frac{f(t)}{F(t)}$$

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$$V'(t) = (V(t) - t) \frac{f(t)}{F(t)} \quad V(1) = 0$$

► Solution:

$$V(t) = t - F(t) \left(1 - \int_t^1 \frac{du}{F(u)} \right)$$

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► Solution:

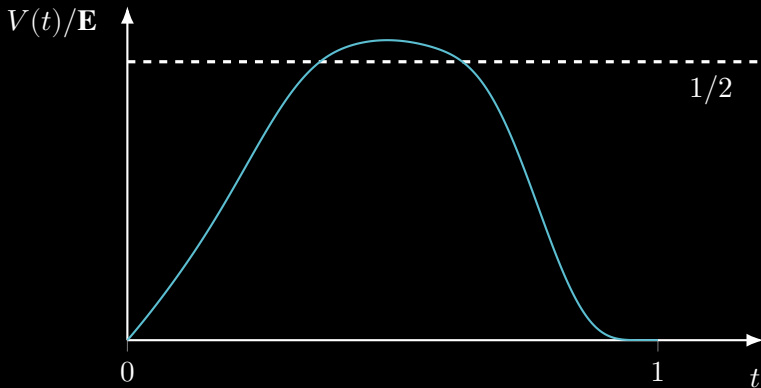
$$V(t) = t - F(t) \left(1 - \int_t^1 \frac{du}{F(u)} \right)$$

► At optimality

$$V(t^*) = t^* \implies \int_{t^*}^1 \frac{du}{F(u)} = 1$$

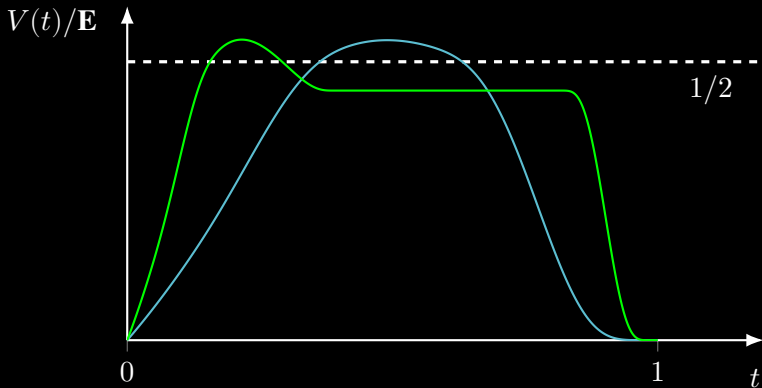
Example: Worst-Case

$$X_1 = \varepsilon \quad X_2 = \begin{cases} 1 & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1 - \varepsilon \end{cases}$$



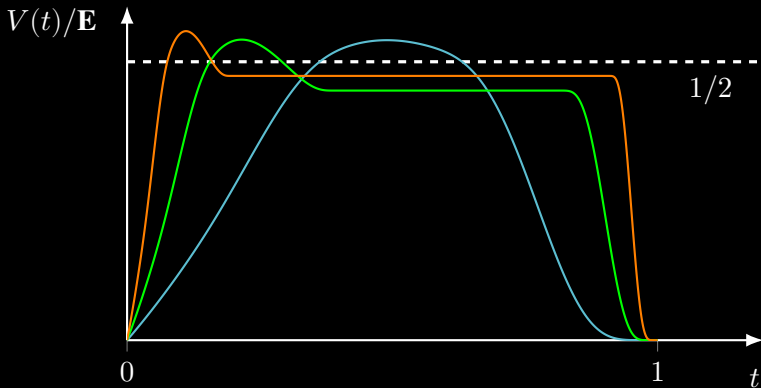
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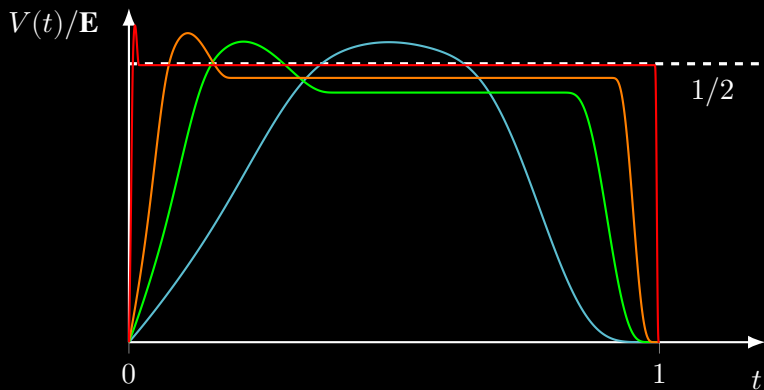
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New Approach: ODE – Results

Let $\mathbf{E} = \int_0^1 (1 - F(u)) du$

From $V'(t) = (V(t) - t) \frac{f(t)}{F(t)}$ we can show:

1. Balanced Prices Lower Bound:

$$V(t) \geq t(1 - F(t)) + F(t)(\mathbf{E} - t)$$

$\Rightarrow$ Both $t = F^{-1}(1/2)$ and $t = \mathbf{E}/2$ give $V(t) \geq \mathbf{E}/2$

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2. Single Sample:

$$\int_0^1 V(t) f(t) dt \geq \frac{\mathbf{E}}{2}$$

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From $V'(t) = (V(t) - t) \frac{f(t)}{F(t)}$ we can show:

3. V is Quasiconcave:

Any t between two “good” t_1, t_2 gives $V(t) \geq \mathbf{E}/2$.

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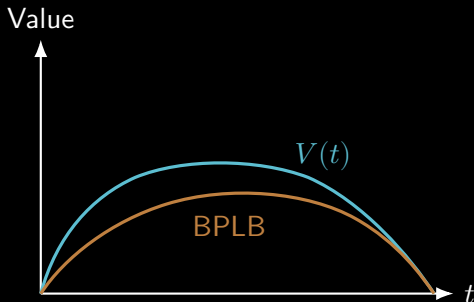
4. Other Results:

► Difference PI: $\mathbf{E} - V(t^*) \leq 1/4$

► k -unit: $\frac{V(t^*)}{\mathbf{E}} \geq 1 - O\left(\sqrt{\frac{\log k}{k}}\right)$

Back to Common Base

- ▶ Recall BPLB gives (analysis-tight) 0.4 for Common Base
- ▶ Using ODE we can get 0.41
- ▶ Cannot get $0.4765 + \varepsilon$ with single-threshold*



Common Scale Model

Observe $Y_i = X_i \cdot Z$

No algorithm can achieve

$$\frac{\mathbb{E}[\text{ALG}]}{\mathbb{E}[\max_i Y_i]} \geq \frac{1}{n} + \varepsilon$$

- ▶ Support of $Z = \{L^0, L^1, \dots, L^i, \dots\}$ for large L
- ▶ We reduce Common Scale to “guessing” the realized exponent of Z

Thank You!

Questions?

