Prophet Inequalities with Oracle Calls and why they're useful

Vasilis Livanos

University of Chile

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Prophet Inequality

 $\text{Observe realizations} \quad X_1, X_2, \ldots, X_n \sim \text{(known)} \ \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n.$

- **Order** is *adversarial*.
- Design algorithm to maximize selected value.
- Compare against all-knowing *prophet*.

$$\blacktriangleright \ \text{If} \ \mathcal{D}_1 = \mathcal{D}_2 = \dots = \mathcal{D}_n \implies \text{IID setting}.$$

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Competitive Ratio (CR or "Prophet Objective"):

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Probability of Selecting Maximum Realization (PMax or "Secretary Objective"):

$$\Pr\left[ALG \text{ selects } \max_i X_i\right]$$

Prophet Objective [Krengel, Sucheston, Garling '77, '78]

 \exists stopping strategy that achieves ${}^1\!/{}_2 \cdot \mathop{\mathbb{E}}[\max_i X_i]$, and this is tight.

Secretary Objective [Esfandiari, HajiAghayi, Lucier, Mitzenmacher '20]

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CR: $T = \frac{\mathbb{E}[\max_i X_i]}{2}$ [Kleinberg, Weinberg '12]
PMax: $\Pr[\max_i X_i \ge T] = 1 - \frac{1}{e}$
[Esfandiari, HajiAghayi, Lucier, Mitzenmacher '20]

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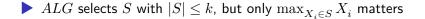
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ALG selects S with $|S| \le k$, but only $\max_{X_i \in S} X_i$ matters

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2018: $1 - e^{-k/6} \leq \mathsf{CR} \leq 1 - k^{-2k}$ for Non-IID. [Ezra, Feldman, Nehama '18]

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Motivation: Auctions/Hiring with overbooking.

 $\mathcal{O}_k\colon$ Assume ALG has k calls to $\mathcal{O},$ who knows $X_1,\ldots,X_n.$

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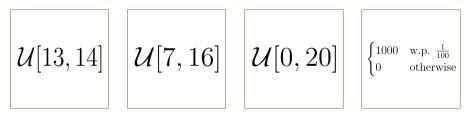
- **Step** *i*:
 - $X_i \ge \max_{j=i+1}^n X_j \Longrightarrow ALG \text{ selects } X_i$
 - $\blacktriangledown X_i < \max_{j=i+1}^n X_j \Longrightarrow ALG \text{ rejects } X_i$
- Generalization of standard PI.
- Allows for simpler analysis since ALG always selects one value.
- "Algorithms with predictions".

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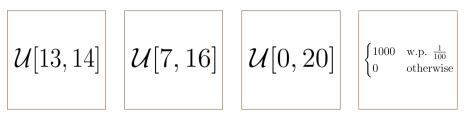
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 - Generalization of standard PI.
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 - A.: YES and NO.

k = 1

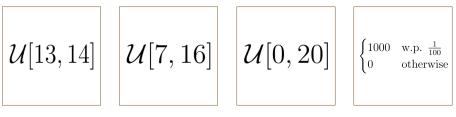


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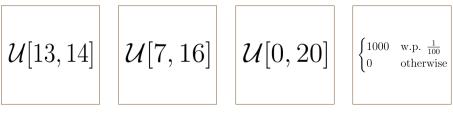
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 $X_1 = 13.67$ $X_2 = 8.59$

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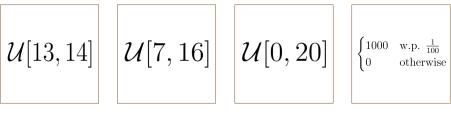


 $X_1 = 13.67$.

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 $X_1 = 13.67 \qquad \qquad X_2 = 8.59 \qquad \qquad X_3 = 2.82 \qquad \qquad X_4 = 0$

$$\begin{split} & \mathbb{E}[\max{\{X_1, X_2, X_3, X_4\}}] \approx 24.66 \\ & \mathbb{E}[OPTALG{\{X_1, X_2, X_3, X_4\}} \text{ for } k = 0] \approx 13.37 \\ & \mathbb{E}[OPTALG{\{X_1, X_2, X_3, X_4\}} \text{ for } k = 1] \approx 19.9 \end{split}$$

Optimal strategy was to ignore X_1, X_2 and query at X_3 .

$\mathcal{O}_k \equiv \operatorname{Top-1-of-}(k+1)$ for PMax

Assume algorithm ${\mathcal A}$ for Top-1-of-(k+1).

 $\blacktriangleright X_i \to \mathcal{A}.$ If \mathcal{A} selects X_i , we use \mathcal{O} .

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Assume algorithm \mathcal{B} for \mathcal{O}_k .

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$\mathcal{O}_k \not\equiv \text{Top-1-of-}(k+1) \text{ for CR}$

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$$X_1 = 1, \quad X_2 = \begin{cases} 1 + \varepsilon & \text{w.p. } 1/2 - \varepsilon \\ 0 & \text{w.p. } 1/2 + \varepsilon \end{cases}, \quad X_3 = \begin{cases} 1/\varepsilon & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1 - \varepsilon \end{cases}$$

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$$\begin{split} & \mathbb{E}[\max{\{X_1, X_2, X_3\}}] \to 2. \\ & \textbf{Let } \mathcal{A} \text{ for Top-1-of-2 that always selects } X_1 \text{ and } X_3. \\ & \mathbb{E}[\mathcal{A}] \to 2. \end{split}$$

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Let
$$\mathcal{B}$$
 for \mathcal{O}_1 .

If B doesn't query at X₁ ⇒ E[B] = E[max {X₂, X₃}] → ³/₂.
 If B queries at X₁ ⇒

$$\mathbb{E}[\mathcal{B}] = \left(\frac{1}{2} + \varepsilon\right)(1 - \varepsilon) \cdot 1 + \left(\frac{1}{2} - \varepsilon\right) \cdot (1 + \varepsilon) + \left(\frac{1}{2} + \varepsilon\right)\varepsilon \cdot \frac{1}{\varepsilon} \to \frac{3}{2}.$$

Theorem [Har-Peled, Harb, L. '24] Single-threshold* algorithm \mathcal{A} achieves

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Almost tight bound, up to the exponent.

New upper bound. Asymptotical improvement on lower bound of [Gilbert, Mosteller '66]

• Idea: Set threshold T such that $Pr[X_i \ge T] = L$ for some L.

Use Chernoff bound to argue that $\leq k$ of the X_i 's are above T and we don't run out of oracle calls.

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- <u>Better idea</u>: Set threshold T such that $\Pr[X_i \ge T] = \frac{e^{\sqrt{k}}}{n}$.

Use Chernoff bound to argue that MAX of sequence of X_i 's above T changes $\leq k$ times \implies We don't run out of queries.

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• Upper Bound: $\mathcal{U}[0,1]$ with roughly $k \log k$ of the X_i 's in $\overline{[1-k\log k/n,1]}$.

Every algorithm will either miss $\max_i X_i$ or run out of oracle calls w.p. $k^{-k}.$

Theorem [Har-Peled, Harb, L. '24]

 \exists sequence $\left\{\xi_k\right\}_{k\in\mathbb{N}}$ and single-threshold* algorithm $\mathcal A$ such that

$$\mathsf{CR}(\mathcal{A}) \geq 1 - \mathcal{O}\left(e^{-\xi_k}\right) = 1 - \mathcal{O}\left(e^{-k/e + \mathcal{O}(k)}\right).$$

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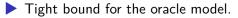
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Asymptotical improvement on both upper and lower bounds of [Ezra, Feldman, Nehama '18].

$$\blacktriangleright \ 1 - e^{-\xi_k} = e^{-\xi_k} \sum_{i=0}^k \frac{(\xi_k)^i}{i!} \iff e^{-\xi_k} = e^{-\xi_k} \sum_{i=k+1}^\infty \frac{(\xi_k)^i}{i!}$$

 ξ_k : Exponent sequence.

Intuition: Balances $\Pr[\mathsf{Poi}(\xi_k) \ge k+1] = \Pr[\mathsf{Poi}(\xi_k) = 0].$

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• Idea: Set threshold T such that $\Pr[\max_i X_i \ge T] = 1 - e^{-\xi_k}$.

For $x \ge T$, analyze $\Pr[ALG \ge x]$ using Poissonization technique of [Harb '23].

$$\blacktriangleright \ 1 - e^{-\xi_k} = e^{-\xi_k} \sum_{i=0}^k \frac{(\xi_k)^i}{i!} \iff e^{-\xi_k} = e^{-\xi_k} \sum_{i=k+1}^\infty \frac{(\xi_k)^i}{i!}$$

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For $x \ge T$, analyze $\Pr[ALG \ge x]$ using Poissonization technique of [Harb '23].

Like in standard PI, we set $Pr[\max_i X_i \ge T] = p$ and get a CR of p.

Upper Bound:

$$X_1 = 1, \quad X_2, \dots, X_{n-1} \sim \mathsf{Poi}(\xi_k), \quad X_n = \begin{cases} 1/\varepsilon & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1 - \varepsilon \end{cases}$$

Upper Bound:

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 $= 2\left(1 - e^{-\xi_k}\right),$

by the choice of ξ_k .

Open Questions

Top-1-of-k Model:

Optimal asymptotics for PMax in IID setting and CR in Non-IID setting (requires new approach).

▶ PMax in Non-IID?

Analyzing oracle model is enough!

CR in IID?

Oracle Model:

lncorrect predictions \implies Robustness, ties with ML.

Questions?

