Prophet Inequalities via Extreme Value Theory
and other short stories

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IID Prophet Inequality

Observe realizations
$X_1, X_2, \dots, X_n \sim$ (known) $\mathcal{D}$.

- Design algorithm to maximize/minimize selected value.
- Compare against all-knowing *prophet*.
- Minimization $\implies$ forced to select some X_i.

Objectives:

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$$\inf\left\{c \,\middle|\, \mathbb{E}[ALG(c\,n)] \geq \mathbb{E}[\max_{i=1}^{n} X_i]\right\} \quad \Bigg| \quad \inf\left\{c \,\middle|\, \mathbb{E}[ALG(c\,n)] \leq \mathbb{E}[\min_{i=1}^{n} X_i]\right\}$$

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[Hill-Kertz '82,
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For any $\mathcal{D}$, $\exists$ threshold stopping strategy $\tau_1, \tau_2, \ldots, \tau_n$ that achieves $\beta \cdot \mathbb{E}[\max_i X_i]$, where $\beta \approx 0.745$, and this is tight.

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Can get something if you allow
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- Competitive Ratio:

No bound on competitive ratio for general (non-I.I.D.) distributions.

[Esfandiari, Hajiaghayi, Liaghat, Monemizadeh '15]

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3. What is ~~the "right"~~ a good way/*tool* to think about IID prophet inequality?

1. No hope for universal bound: [Lucier '22]

   $\mathcal{D} : F(x) = 1 - 1/x$, with $x \in [1, +\infty)$ (Equal-revenue distribution).
   $\mathbb{E}[X] = 1 + \int_1^\infty (1 - F(x))\, dx = +\infty$, but

   $\mathbb{E}[\min\{X_1, X_2\}] = 1 + \int_1^\infty (1 - F(x))^2\, dx < +\infty.$

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Fix $\mathcal{D}$ and take $n \to \infty$.

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- $\lim_{n\to\infty} M_n = +\infty, \quad \lim_{n\to\infty} m_n = 0 \implies$ Re-scaling

Main Tool: Extreme Value Theory

Extreme Value Theorem [Fisher, Tippett '28, Gnedenko '43]

Assume there exist sequences $a_n > 0, b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} F_{M_n}(a_n x + b_n) = G_\gamma^+(x).$$

Then,

$$G_\gamma^+(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), & \text{if } \gamma \neq 0 \\ \exp\left(-\exp\left(-x\right)\right), & \text{if } \gamma = 0 \end{cases}.$$

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- Three distinct G_γ^+'s:
  - $\gamma < 0$: Reverse Weibull
  - $\gamma = 0$: Gumbel
  - $\gamma > 0$: Fréchet

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- Can get similar result for $\textsc{Min}$, but γ changes.
- Conditions $\implies \mathcal{D}$ follows EVT.

IID PI via Extreme Value Theory

$$\boxed{\Gamma(x) = (x-1)!}$$

Theorem

Assume there exist sequences $a_n > 0, b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} F_{M_n}(a_n x + b_n) = G_\gamma^+(x)$$

for some γ

$$\lim_{n\to\infty} F_{m_n}(a_n x + b_n) = G_\gamma^-(x)$$

for some γ

Then, the optimal DP achieves a competitive ratio, as $n \to \infty$, of

$$ACR_{Max} = \min\left\{\frac{(1-\gamma)^{-\gamma}}{\Gamma\left(1-\gamma\right)}, 1\right\}.$$

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- Unified analysis of competitive ratio for both $\textsc{Max}$ and $\textsc{Min}$.

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- $\mathcal{D}$ MHR $\implies \quad ACR_{Max} = 1 \quad \& \quad ACR_{Min} \le 2$

Single-Threshold Algorithm?

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for some γ. Then, the optimal single-threshold algorithm T for minimization achieves $ACR_{Min}(T) = \mathrm{O}\left((\log n)^{-\gamma}\right)$.

I lied*!

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Maximization

[Kennedy, Kertz '91]]

For any $\mathcal{D}$ following EVT, the asymptotic competitive ratio of the optimal DP is

- 1 for $\gamma \leq 0$
- $\frac{(1-\gamma)^{-\gamma}}{\Gamma(1-\gamma)}$ for $\gamma \in (0, 1)$
- 1 for $\gamma \geq 1$ (as $\mathbb{E}[X] = +\infty$)

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Maximization

[Hill, Kertz '82]]

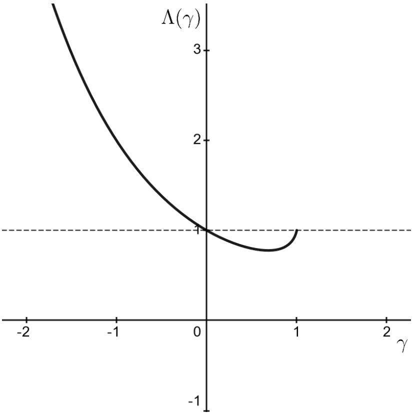
For any $\mathcal{D}$, $\exists$ a *single threshold* τ such that selecting the first $X_i \geq \tau$ achieves $(1 - {}^1\!/_e) \cdot \mathbb{E}[\max_i X_i] \approx 0.632 \cdot \mathbb{E}[\max_i X_i]$.

[Correa, Pizarro, Verdugo '21]]

For any $\mathcal{D}$ following EVT, the asymptotic competitive ratio of the optimal *single threshold* algorithm is

- 1 for $\gamma \leq 0$
- $\frac{-(-\gamma - W_{-1}(-\gamma\, e^{-\gamma}))^{1-\gamma}}{(1-\gamma)\Gamma(1-\gamma)W_{-1}(-\gamma\, e^{-\gamma})}$ for $\gamma \in (0, 1)$
- 1 for $\gamma \geq 1$ (as $\mathbb{E}[X] \to +\infty$)

Asymptotic Competitive Ratio



For $\gamma \to -\infty$, by Stirling's approximation

$$\frac{(1-\gamma)^{-\gamma}}{\Gamma(1-\gamma)} \approx e^{-\gamma}.$$

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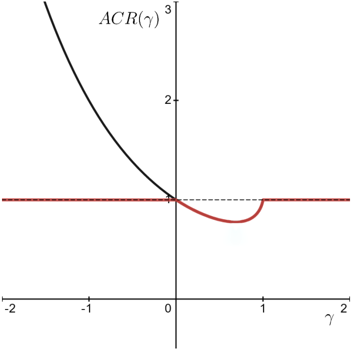

Figure: ACR(γ) for MAX

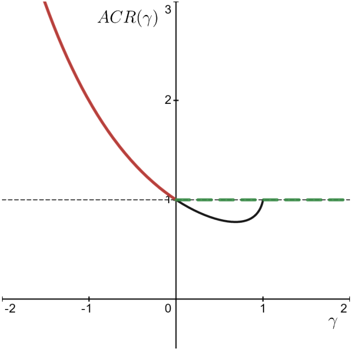

Figure: ACR(γ) for MIN

High-Level Approach

$F(t) = \Pr_{X\sim\mathcal{D}}[X \le t]$, $\quad F^{\leftarrow}(p)$: inverse of F ("Quantile function").

Using EVT and heavy-machinery from theory of regularly-varying functions:

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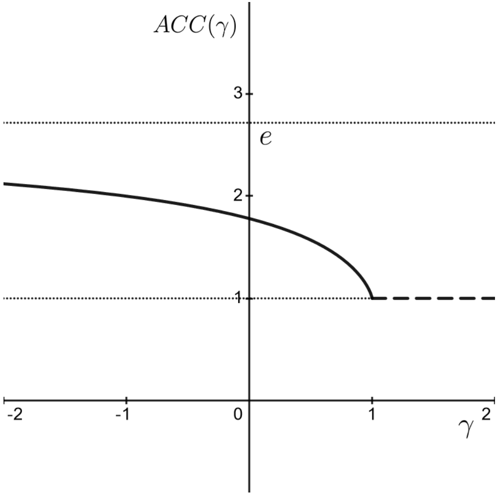

- $ACC \leq e$ for all $\mathcal{D}$ following EVT.

Questions?